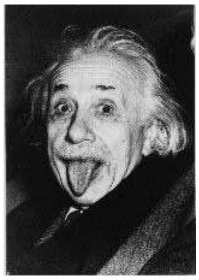

Lecture Notes on General Relativity

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These notes are also available from

`http://www.blau.itp.unibe.ch/Lecturenotes.html`

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Contents

0.1 Introduction

The year 1905 was Einstein's magical year. In that year, he published three articles, on light quanta, on the foundations of the theory of Special Relativity, and on Brownian motion, each one separately worthy of a Nobel prize. Immediately after his work on Special Relativity, Einstein started thinking about gravity and how to give it a relativistically invariant formulation. He kept on working on this problem during the next ten years, doing little else. This work, after many trials and errors, culminated in his masterpiece, the *General Theory of Relativity*, presented in 1915/1916. It is clearly one of the greatest scientific achievements of all time, a beautiful theory derived from pure thought and physical intuition, capable of explaining, still today, almost 100 years later, virtually every aspect of gravitational physics ever observed.

Einstein's key insight was that gravity is not a physical external force like the other forces of nature but rather a manifestation of the *curvature of space-time* itself. This realisation, in its simplicity and beauty, has had a profound impact on theoretical physics as a whole, and Einstein's vision of a geometrisation of all of physics is still with us today.

Of course, we do not have ten years to reach these insights but nevertheless the first half of this course will be dedicated to explaining this and to developing the machinery (of tensor calculus and Riemannian geometry) required to describe physics in a curved space time, i.e. in a gravitational field.

In the second half of this course, we will then turn to various applications of General Relativity. Foremost among them is the description of the classical predictions of General Relativity and their experimental verification. Other subjects we will cover include the strange world of Black Holes, Cosmology, gravitational waves, and some intriguing theories of gravity in higher dimensions known as Kaluza-Klein theories.

General Relativity may appear to you to be a difficult subject at first, since it requires a certain amount of new mathematics and takes place in an unfamiliar arena. However, this course is meant to be essentially self-contained, requiring only a basic familiarity with Special Relativity, vector calculus and coordinate transformations. That means that I will attempt to explain every single other thing that is required to understand the basics of Einstein's theory of gravity.

0.2 Caveats and Omissions

Invariably, any set of (introductory) lecture notes has its shortcomings, due to lack of space and time, the requirements of the audience and the expertise (or lack thereof) of the lecturer. These lecture notes are, of course, no exception.

These lecture notes for an introductory course on General Relativity are based on a course that I originally gave in the years 1998-2003 in the framework of the Diploma

Course of the ICTP (Trieste, Italy). Currently these notes form the basis of a course that I teach as part of the Master in Theoretical Physics curriculum at the University of Bern.

The purpose of these notes is to supplement the course, not to replace a text-book. You should turn to other sources for other explanations, more on the historical and experimental side, and exercises to test your understanding of these matters. Nevertheless, I hope that these notes are reasonably self-contained and comprehensible.

I make no claim to originality in these notes. In particular, the presentation of much of the introductory material follows quite closely the treatment in Weinberg's classic

- S. Weinberg, *Gravitation and Cosmology*

(and I have made no attempt to disguise this). Even though my own way of thinking about general relativity is somewhat more geometric, I have found that the approach adopted by Weinberg is ideally suited to introduce general relativity to students with little mathematical background. I have also used a number of other sources, including, in particular, Sean Carroll's on-line lecture notes

- `http://pancake.uchicago.edu/~carroll/notes/`

which have, in the meantime, been expanded into the lovely textbook

- S. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*

Other books that I like and recommend are

- J. Hartle, *Gravity: an Introduction to Einstein's General Relativity*
- E. Poisson, *A Relativist's Toolkit: the Mathematics of Black Hole Mechanics*
- R. Wald, *General Relativity*

Sections marked with a * contain supplementary material that lies somewhat outside the scope of a standard introductory course on general relativity and/or is not strictly necessary for an understanding of the subsequent sections.

Further additions and updates to these notes are in preparation. I am grateful for feedback of any kind: complaints, constructive criticism, corrections, and suggestions for what else to include. If you have any comments on these notes, or also if you just happen to find them useful, please let me know (`blau@itp.unibe.ch`).

I believe/hope that the strengths of these lecture notes are that

- they are elementary, requiring nothing beyond special relativity and calculus [to be precise, by special relativity I mean the covariant formulation in terms of the Minkowski metric etc.; special relativity is (regardless of what you may have been taught) not fundamentally a theory about people changing trains erratically, running into barns with poles, or doing strange things to their twins; rather, it is a theory of a fundamental symmetry principle of physics, namely that the laws of physics are invariant under Lorentz transformations and that they should therefore also be formulated in a way which makes this symmetry manifest.]

- they are essentially self-contained,

- they provide a balanced overview of the subject, the second half of the course dealing with a larger variety of different subjects than is usually covered in a 20 lecture introductory course.

In my opinion, among the weaknesses of this course or these lecture notes are the following:

- The history of the development of general relativity is an important and complex subject, crucial for a thorough appreciation of general relativity. My remarks on this subject are scarce and possibly even misleading at times and should not be taken as gospel.

- Exercises are an essential part of the course, but so far I have not included them in the lecture notes themselves.

- In the first half of the course, on tensor calculus, no mention is made of manifolds and bundles as this would require some background in differential geometry and topology I did not want to assume.

- Moreover, practically no mention is made of the manifestly coordinate independent calculus of differential forms. Given a little bit more time, it would be possible to cover the (extremely useful) vielbein and differential form formulations of general relativity, and a supplement to these lecture notes on this subject is in preparation.

- The discussion of the causal structure of the Schwarzschild metric and its Kruskal-Szekeres extension stops short of introducing Penrose diagrams. These are useful and important and, once again, given a bit more time, this is a subject that could and ought to be covered as well.

- Cosmology is a very active, exciting, and rapidly developing field. Unfortunately, not being an expert on the subject, my treatment is rather old-fashioned and certainly not Y2K compatible. I would be grateful for suggestions how to improve this section.

- Something crucial is missing from the section on gravitational waves, namely a derivation of the famous quadrupole radiation formula. If I can come up with, or somebody shares with me, a simple five-line derivation of this formula, I will immediately include it here.

- There are numerous other important topics not treated in these notes, foremost among them perhaps a discussion of the canonical ADM formalism, a discussion of notions of energy in general relativity, the post-Newtonian approximation, other exact solutions, and aspects of black hole thermodynamics.

Including all these topics would require at least one more one-semester course and would turn these lecture notes into a (rather voluminous) book. The former was not possible, given initially the constraints of the ICTP Diploma Course and now those of the Bologna system, and the latter is not my intention, since a number of excellent textbooks on General Relativity have appeared in recent years. I can only hope that these lecture notes provide the necessary background for studying these more advanced topics.

Part I: Towards the Einstein Equations

1 From the Einstein Equivalence Principle to Geodesics

1.1 Motivation: The Einstein Equivalence Principle

First of all, let us ask the question why we should not be happy with the classical Newtonian description of gravity. Well, for one, this theory is not Lorentz invariant, postulating an action at a distance and an instantaneous propagation of the gravitational field to every point in space. This is something that Einstein had just successfully exorcised from other aspects of physics, and clearly Newtonian gravity had to be revised as well.

It is then immediately clear that what would have to replace Newton's theory is something rather more complicated. The reason for this is that, according to Special Relativity, mass is just another form of energy. But then, since gravity couples to masses, in a relativistically invariant theory, gravity will also couple to energy. In particular, therefore, gravity would have to couple to gravitational energy, i.e. to itself. As a consequence, the new gravitational field equations will, unlike Newton's, have to be non-linear: the field of the sum of two masses cannot equal the sum of the gravitational fields of the two masses because it should also take into account the gravitational energy of the two-body system.

But now, having realised that Newton's theory cannot be the final word on the issue, how does one go about finding a better theory?

I will first very briefly discuss (and then dismiss) what at first sight may appear to be the most natural and simple approach to formulating a relativistic theory of gravity, namely the simple replacement of Newton's field equation

$$\Delta\phi = 4\pi G\rho \tag{1.1}$$

for the gravitational potential ϕ given a mass density ρ by its relativistically covariant version

$$\Delta\phi = 4\pi G\rho \quad \longrightarrow \quad \Box\phi = 4\pi G\rho \ . \tag{1.2}$$

While this looks promising, something can't be quite right about this equation. We already know (from Special Relativity) that ρ is not a scalar but rather the 00-component of a tensor, the energy-momentum tensor, so if actually ρ appears on the right-hand-side, ϕ cannot be a scalar, while if ϕ is a scalar something needs to be done to fix the right-hand-side.

Turning first to the latter possiblility, one option that suggests itself is to replace ρ by the trace $T = T^{\alpha}_{\ \alpha}$ of the energy-momentum tensor. This is by definition / construction a scalar, and it will agree with ρ in the non-relativistic limit. Thus a first attempt at fixing the above equation might look like

$$\Box\phi = 4\pi G T \quad . \tag{1.3}$$

This is certainly an attractive equation, but it definitely has the drawback that it is too linear. Recall from the discussion above that the universality of gravity (coupling to all forms of matter) and the equivalence of mass and energy lead to the conclusion that gravity should couple to gravitational energy, invariably predicting non-linear (self-interacting) equations for the gravitational field. However, the left hand side could be such that it only reduces to $\Box$ or Δ of the Newtonian potential in the Newtonian limit of weak stationary fields. Thus a second attempt at fixing the above equation might look like

$$\Box\Phi(\phi) = 4\pi G T \quad , \tag{1.4}$$

where $\Phi(\phi) \approx \phi$ for weak fields.

Such a scalar relativistic theory of gravity and variants thereof were proposed and studied among others by Abraham, Mie, and Nordstrøm. As it stands, this field equation appears to be perfectly consistent (and it may be interesting to discuss if/how the Einstein equivalence principle, which will put us on our route towards metrics and space-time curvature) is realised in such a theory. However, regardless of this, this theory is incorrect simply because it is ruled out experimentally. The easiest way to see this (with hindsight) is to note that the energy-momentum tensor of Maxwell theory (5.28) is traceless, and thus the above equation would predict no coupling of gravity to the electro-magnetic field, in particular to light, hence in such a theory there would be no deflection of light by the sun etc.[1]

The other possibility to render (1.2) consistent is the, a priori perhaps much less compelling, option to think of ϕ and $\Delta\phi$ or $\Box\phi$ not as scalars but as (00)-components of some tensor, in which case one could try to salvage (1.2) by promoting it to a tensorial equation

$$\{\text{Some tensor generalising } \Delta\phi\}_{\alpha\beta} \sim 4\pi G T_{\alpha\beta} \quad . \tag{1.5}$$

This is indeed the form of the field equations for gravity (the Einstein equations) we will ultimately be led to (see section 9.4), but Einstein arrived at this in a completely different, and much more insightful, way.

Let us now, very briefly and in a streamlined way, try to retrace Einstein's thoughts which, as we will see, will lead us rather quickly to the geometric picture of gravity sketched in the Introduction. He approached this by thinking about three related issues,

1. the equivalence principle of Special Relativity;

[1] For more on the history and properties of scalar theories of gravity see the review by D. Giulini, *What is (not) wrong with scalar gravity?*, `arXiv:gr-qc/0611100`.

2. the relation between inertial and gravitational mass;
3. Special Relativity and accelerations.

As regards the first issue, let me just recall that Special Relativity postulates a preferred class of inertial frames, namely those travelling at constant velocity to each other. But this raises the questions (I will just raise and not attempt to answer) what is special about constant velocities and, more fundamentally, velocities constant with respect to what? Some absolute space? The background of the stars? . . . ?

Regarding the second issue, recall that in Newtonian theory, classical mechanics, there are two a priori independent concepts of mass: inertial mass m_i, which accounts for the resistance against acceleration, and gravitational mass m_g which is the mass gravity couples to. Now it is an important empirical fact that the inertial mass of a body is equal to its gravitational mass. This is usually paraphrased as 'all bodies fall at the same rate in a gravitational field'. This realisation, at least with this clarity, is usually attributed to Galileo (it is not true, though, that Galileo dropped objects from the leaning tower of Pisa to test this - he used an inclined plane, a water clock and a pendulum).

These experiments were later on improved, in various forms, by Huygens, Newton, Bessel and others and reached unprecedented accuracy with the work of Baron von Eötvös (1889-. . .), who was able to show that inertial and gravitational mass of different materials (like wood and platinum) agree to one part in 10^9. In the 1950/60's, this was still further improved by R. Dicke to something like one part in 10^{11}. More recently, rumours of a 'fifth force', based on a reanalysis of Eötvös' data (but buried in the meantime) motivated experiments with even higher accuracy and no difference between m_i and m_g was found.

Now Newton's theory is in principle perfectly consistent with $m_i \neq m_g$, and Einstein was very impressed with their observed equality. This should, he reasoned, not be a mere coincidence but is probably trying to tell us something rather deep about the nature of gravity. With his unequalled talent for discovering profound truths in simple observations, he concluded (calling this "der glücklichste Gedanke meines Lebens" (the happiest thought of my life)) that the equality of inertial and gravitational mass suggests a close relation between inertia and gravity itself, suggests, in fact, that locally effects of gravity and acceleration are indistinguishable,

$$\text{locally: GRAVITY} = \text{INERTIA} = \text{ACCELERATION}$$

He substantiated this with some classical thought experiments, *Gedankenexperimente*, as he called them, which have come to be known as the *elevator thought experiments*:

1. Consider somebody in a small sealed box (elevator) somewhere in outer space. In

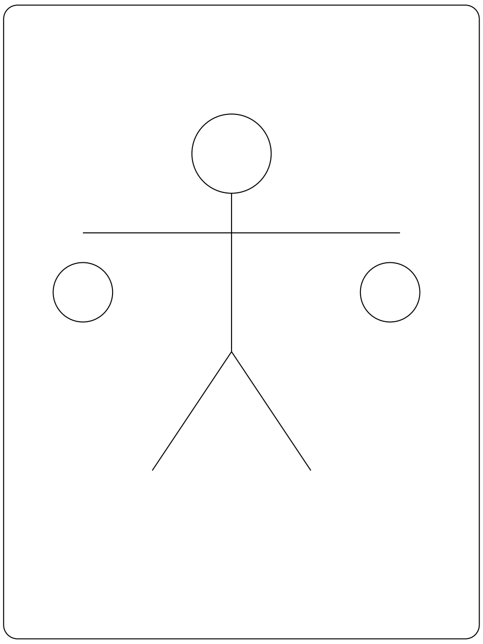

Figure 1: An experimenter and his two stones freely floating somewhere in outer space, i.e. in the absence of forces.

the absence of any forces, this person will float. Likewise, two stones he has just dropped (see Figure 1) will float with him.

2. Now assume (Figure 2) that somebody on the outside suddenly pulls the box up with a constant acceleration. Then of course, our friend will be pressed to the bottom of the elevator with a constant force and he will also see his stones drop to the floor.

3. Now consider (Figure 3) this same box brought into a constant gravitational field. Then again, he will be pressed to the bottom of the elevator with a constant force and he will see his stones drop to the floor. With no experiment inside the elevator can he decide if this is actually due to a gravitational field or due to the fact that somebody is pulling the elevator upwards.

   Thus our first lesson is that, indeed, locally the effects of acceleration and gravity are indistinguishable.

4. Now consider somebody cutting the cable of the elevator (Figure 4). Then the elevator will fall freely downwards but, as in Figure 1, our experimenter and his stones will float as in the absence of gravity.

   Thus lesson number two is that, locally the effect of gravity can be eliminated by going to a freely falling reference frame (or coordinate system). This should not come as a surprise. In the Newtonian theory, if the free fall in a constant

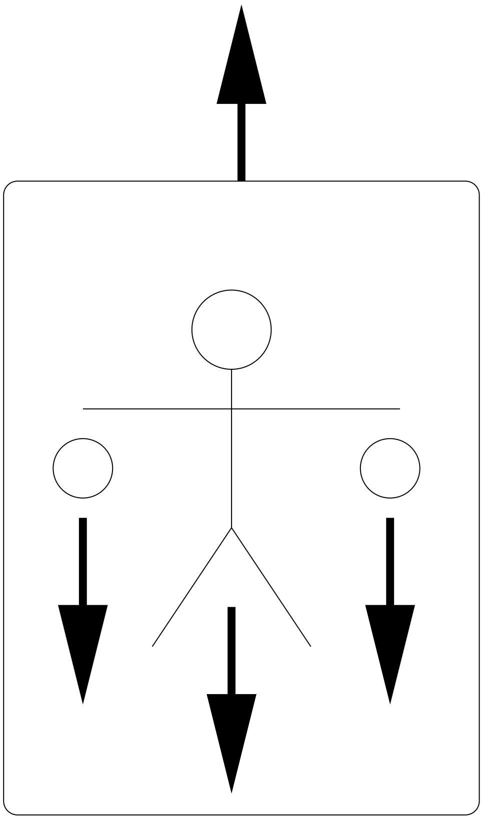

Figure 2: Constant acceleration upwards mimics the effect of a gravitational field: experimenter and stones drop to the bottom of the box.

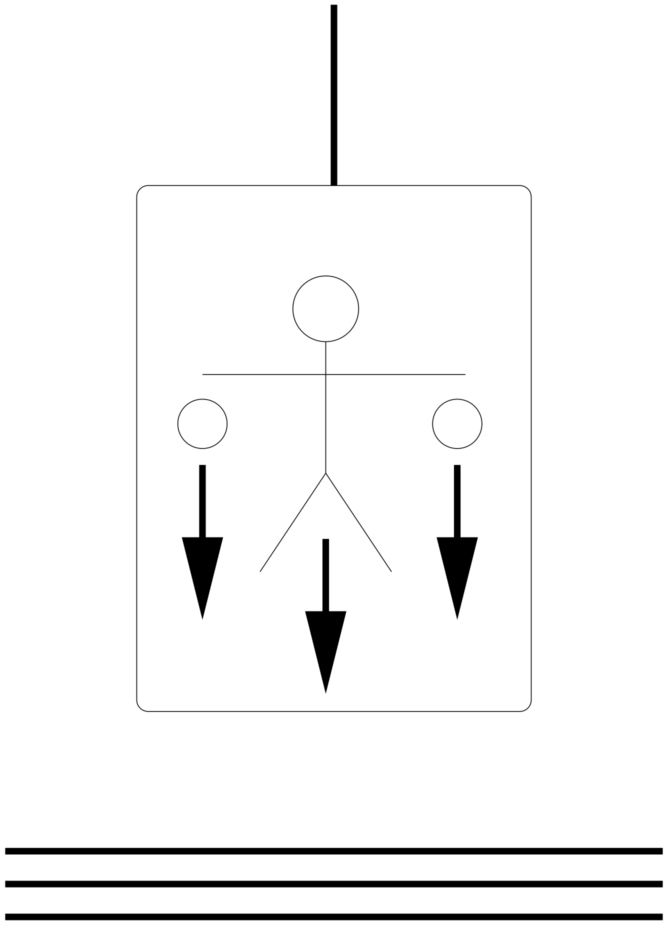

Figure 3: The effect of a constant gravitational field: indistinguishable for our experimenter from that of a constant acceleration in Figure 2.

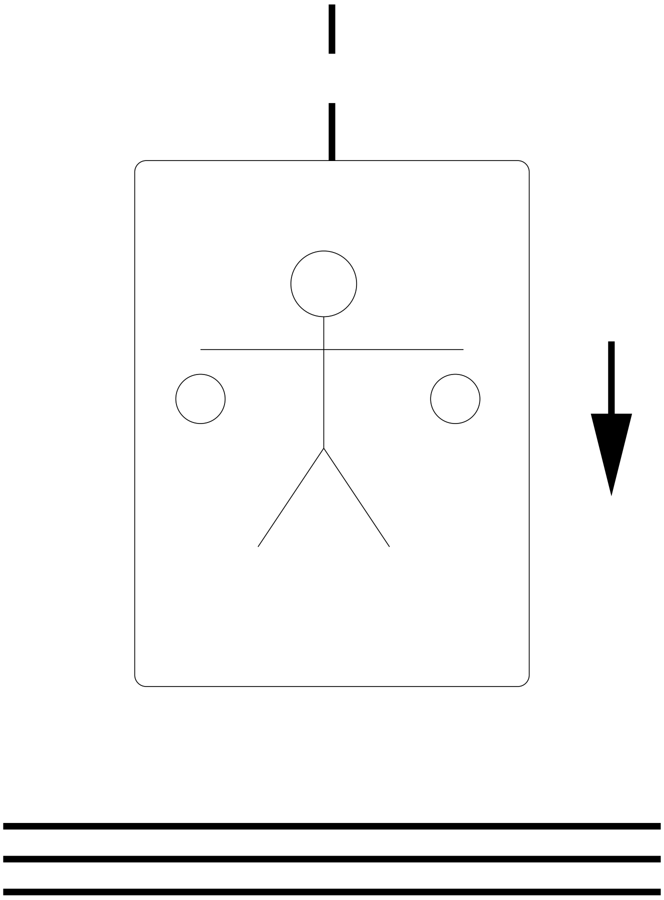

Figure 4: Free fall in a gravitational field has the same effect as no gravitational field (Figure 1): experimenter and stones float.

gravitational field is described by the equation

$$\ddot{x} = g \;\; (+ \;\text{ other forces}) \;\;, \tag{1.6}$$

then in the accelerated coordinate system

$$\xi(x,t) = x - gt^2/2 \tag{1.7}$$

the same physics is described by the equation

$$\ddot{\xi} = 0 \;\; (+ \;\text{ other forces}) \;\;, \tag{1.8}$$

and the effect of gravity has been eliminated by going to the freely falling coordinate system ξ. The crucial point here is that in such a reference frame not only our observer will float freely, but he will also observe all other objects obeying the usual laws of motion in the absence of gravity.

5. In the above discussion, I have put the emphasis on constant accelerations and on 'locally'. To see the significance of this, consider our experimenter with his elevator in the gravitational field of the earth (Figure 5). This gravitational field is not constant but spherically symmetric, pointing towards the center of the earth. Therefore the stones will slightly approach each other as they fall towards the bottom of the elevator, in the direction of the center of the gravitational field.

   Thus, if somebody cuts the cable now and the elevator is again in free fall (Figure 6), our experimenter will float again, so will the stones, but our experimenter will also notice that the stones move closer together for some reason. He will have to conclude that there is some force responsible for this.

   This is lesson number three: in a non-uniform gravitational field the effects of gravity cannot be eliminated by going to a freely falling coordinate system. This is only possible locally, on such scales on which the gravitational field is essentially constant.

Einstein formalised the outcome of these thought experiments in what is now known as the *Einstein Equivalence Principle* which roughly states that physics in a freely falling frame in a gravitational field is the same physics in an inertial frame in Minkowski space in the absence of gravitation. Two formulation are

> At every space-time point in an arbitrary gravitational field it is possible to choose a *locally inertial* (or *freely falling*) *coordinate system* such that, within a sufficiently small region of this point, the laws of nature take the same form as in unaccelerated Cartesian coordinate systems in the absence of gravitation. (Weinberg, Gravitation and Cosmology)

and

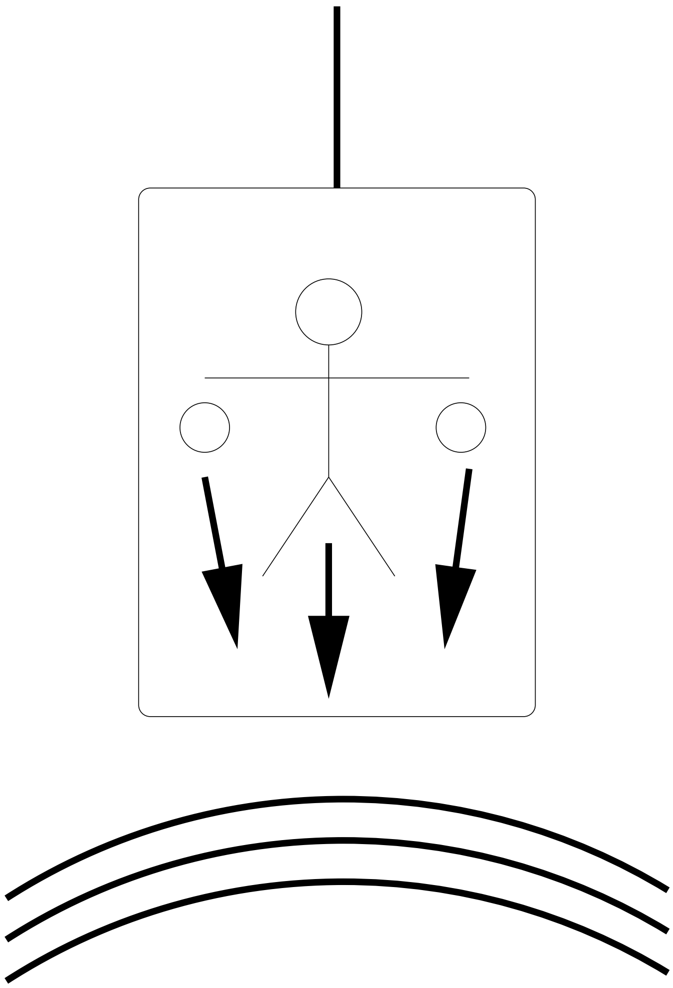

Figure 5: The experimenter and his stones in a non-uniform gravitational field: the stones will approach each other slightly as they fall to the bottom of the elevator.

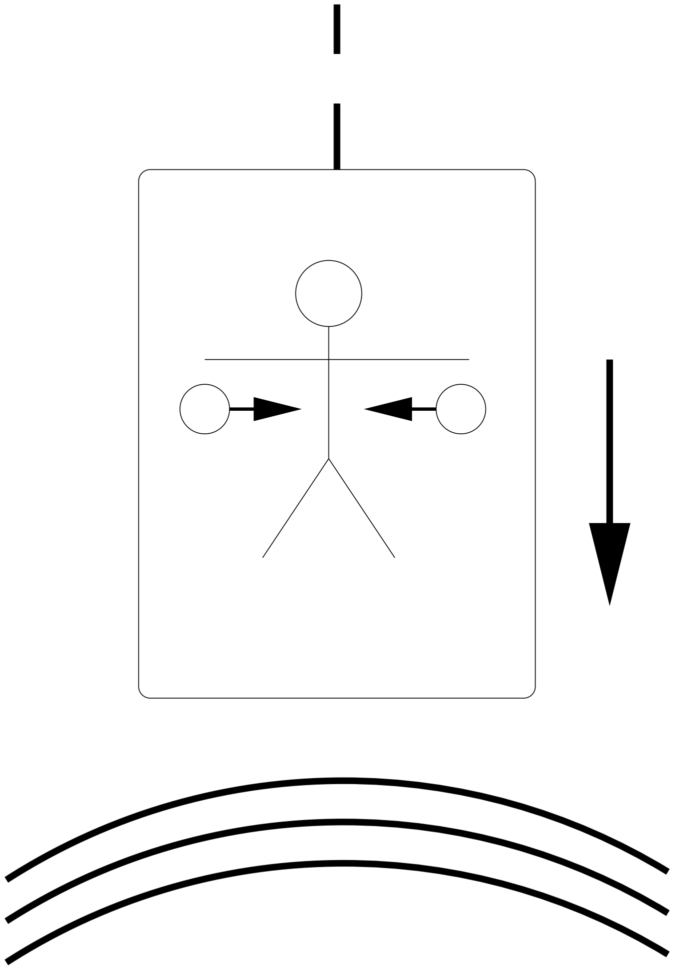

Figure 6: Experimentator and stones freely falling in a non-uniform gravitational field. The experimenter floats, so do the stones, but they move closer together, indicating the presence of some external force.

> Experiments in a sufficiently small freely falling laboratory, over a sufficiently short time, give results that are indistinguishable from those of the same experiments in an inertial frame in empty space. (Hartle, Gravity).

There are different versions of this principle depending on what precisely one means by 'the laws of nature'. If one just means the laws of Newtonian (or relativistic) mechanics, then this priciple essentially reduces to the statement that inertial and gravitational mass are equal. Usually, however, this statement is taken to imply also Maxwell's theory, quantum mechanics etc. What it asserts in its strong form is that

> [. . .] there is no experiment that can distinguish a uniform acceleration from a uniform gravitational field. (Hartle, Gravity)

The power of the above principle lies in the fact that we can combine it with our understanding of physics in accelerated reference systems to gain inisght into the physics in a gravitational field. Two immediate consequences of this (which are hard to derive on the basis of Newtonian physics or Special Relativity alone) are

- light is deflected by a gravitational field just like material objects;
- clocks run slower in a gravitational field than in the absence of gravity.

To see the inevitability of the first assertion, imagine a light ray entering the rocket / elevator in Figure 1 horizontally through a window on the left hand side and exiting again at the same height through a window on the right. Now imagine, as in Figure 2, accelerating the elevator upwards. Then clearly the light ray that enters on the left will exit at a lower point of the elevator on the right because the elevator is accelerating upwards. By the equivalence principle one should observe exactly the same thing in a constant gravitational field (Figure 3). It follows that in a gravitational field the light ray is bent downwards, i.e. it experiences a downward acceleration with the (locally constant) gravitational acceleration g.

To understand the second assertion, one can e.g. simply appeal to the so-called "twin-paradox" of Special Relativity: the accelerated twin is younger than his unaccelerated inertial sibling. Hence accelerated clocks run slower than inertial clocks. Hence, by the equivalence principle, clocks in a gravitational field run slower than clocks in the absence of gravity.

Alternatively, one can imagine two observers at the top and bottom of the elevator, having identical clocks and sending light signals to each other at regular intervals as determined by their clocks. Once the elevator accelerates upwards, the observer at the bottom will receive the signals at a higher rate than he emits them (because he is accelerating towards the signals he receives), and he will interpret this as his clock running more slowly than that of the observer at the top. By the equivalence principle, the same conclusion now applies to two observers at different heights in a gravitational field. This can also be interpreted in terms of a gravitational red-shift or blue-shift (photons losing or gaining energy by climbing or falling in a gravitational field), and we will return to a more quantitative discussion of this effect in section 2.6.

1.2 Accelerated Observers and the Rindler Metric

What the equivalence principle tells us is that we can expect to learn something about the effects of gravitation by transforming the laws of nature (equations of motion) from an inertial Cartesian coordinate system to other (accelerated, curvilinear) coordinates.

As a first step, let us discuss the above example of an observer undergoing constant acceleration in the context of special relativity. This will also serve to set the notation and recall some basic facts regarding the Lorentz-covariant formulation of special relativity.

In the covariant formulation, the timelike worldline of an observer is described by the functions $\xi^A(\tau)$, where ξ^A are standard inertial Minkowski coordinates in terms of which the line element of Minkowski space-time [henceforth *Minkowski space* for short, the union of space and time is implied by "Minkowski"] takes the form

$$ds^2 = \eta_{AB} d\xi^A d\xi^B \quad , \tag{1.9}$$

where $(\eta_{AB}) = \text{diag}(-1, +1, +1, +1)$, and τ is the Lorentz-invariant proper time, defined by

$$d\tau = \sqrt{-\eta_{AB} d\xi^A d\xi^B} \quad . \tag{1.10}$$

It follows that the velocity 4-vector $u^A = d\xi^A/d\tau$ is normalised as

$$u^A u_A \equiv \eta_{AB} u^A u^B = -1 \quad . \tag{1.11}$$

The Lorentz-covariant acceleration is the 4-vector

$$a^A = \frac{d}{d\tau} u^A = \frac{d^2}{d\tau^2} \xi^A \quad , \tag{1.12}$$

and in such an inertial coordinate system the equation of motion of a massive free particle is

$$\frac{d^2}{d\tau^2} \xi^A(\tau) = 0 \quad . \tag{1.13}$$

We will study this equation further in the next subsection. For now we look at observers with non-zero acceleration. It follows from (1.11) by differentiation that a^A is orthogonal to u^A,

$$a^A u_A \equiv \eta_{AB} a^A u^B = 0 \quad , \tag{1.14}$$

and therefore spacelike,

$$\eta_{AB} a^A a^B \equiv \mathsf{a}^2 > 0 \quad . \tag{1.15}$$

Specialising to an observer accelerating in the ξ^1-direction (so that in the momentary restframe of this observer one has $u^A = (1,0,0,0), a^A = (0,\mathsf{a},0,0)$), we will say that the observer undergoes constant acceleration if a is time-independent. To determine the worldline of such an observer, we note that the general solution to (1.11) with $u^2 = u^3 = 0$,

$$\eta_{AB}u^A u^B = -(u^0)^2 + (u^1)^2 = -1 \quad , \tag{1.16}$$

is

$$u^0 = \cosh F(\tau) \quad , \quad u^1 = \sinh F(\tau) \tag{1.17}$$

for some function $F(\tau)$. Thus the acceleration is

$$a^A = \dot{F}(\tau)(\sinh F(\tau), \cosh F)(\tau), 0, 0) \quad , \tag{1.18}$$

with norm

$$\mathsf{a}^2 = \dot{F}^2 \quad , \tag{1.19}$$

and an observer with constant acceleration is characterised by $F(\tau) = \mathsf{a}\tau$,

$$u^A(\tau) = (\cosh \mathsf{a}\tau, \sinh \mathsf{a}\tau, 0, 0) \quad . \tag{1.20}$$

This can now be integrated, and in particular

$$\xi^A(\tau) = (\mathsf{a}^{-1}\sinh \mathsf{a}\tau, \mathsf{a}^{-1}\cosh \mathsf{a}\tau, 0, 0) \tag{1.21}$$

is the worldline of an observer with constant acceleration a and initial condition $\xi^A(\tau = 0) = (0, \mathsf{a}^{-1}, 0, 0)$. The worldlines of this observer is the hyperbola

$$\eta_{AB}\xi^A\xi^B = -(\xi^0)^2 + (\xi^1)^2 = \mathsf{a}^{-2} \tag{1.22}$$

in the quadrant $\xi^1 > |\xi^0|$ of Minkoskwi space-time.

We can now ask the question what the Minkowski metric or line-element looks like in the restframe of such an observer. Note that one cannot expect this to be again the constant Minkowski metric η_{AB}: the transformation to an accelerated reference system, while certainly allowed in special relativity, is not a Lorentz transformation, while η_{AB} is, by definition, invariant under Lorentz-transformations. We are thus looking for coordinates that are adapted to these accelerated observers in the same way that the inertial coordinates are adapted to stationary observers (ξ^0 is proper time, and the spatial components ξ^i remain constant). In other words, we seek a coordinate transformation $(\xi^0, \xi^1) \to (\eta, \rho)$ such that the worldlines of these accelerated observers are characterised by $\rho =$ constant (this is what we mean by restframe, the observer stays at a fixed value of ρ) and ideally such that then η is proportional to the proper time of the observer. Comparison with (1.21) suggests the coordinate transformation

$$\xi^0(\eta, \rho) = \rho \sinh \eta \qquad \xi^1(\eta, \rho) = \rho \cosh \eta \quad . \tag{1.23}$$

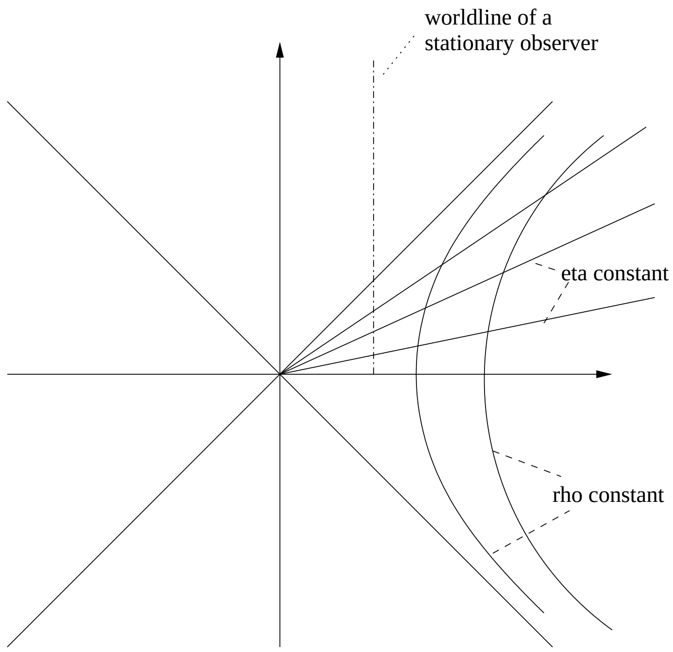

Figure 7: The Rindler metric: Rindler coordinates (η, ρ) cover the first quadrant $\xi^1 > |\xi^0|$. Indicated are lines of constant ρ (hyperbolas, worldlines of constantly accelerating observers) and lines of constant η (straight lines through the origin). The quadrant is bounded by the lightlike lines $\xi^0 = \pm\xi^1 \Leftrightarrow \eta = \pm\infty$. A stationary observer reaches and crosses the line $\eta = \infty$ in finite proper time $\tau = \xi^0$.

It is now easy to see that in terms of these new coordinates the 2-dimensional Minkowski metric $ds^2 = -(d\xi^0)^2 + (d\xi^1)^2$ (we are now suppressing, here and in the remainder of this subesection, the transverse spectator dimensions 2 and 3) takes the form

$$ds^2 = -\rho^2 d\eta^2 + d\rho^2 \quad . \tag{1.24}$$

This is the so-called *Rindler metric*. Let us gain a better understanding of the Rindler coordinates ρ and η, which are obvisouly in some sense hyperbolic (Lorentzian) analogues of polar coordinates ($x = r\cos\phi, y = r\sin\phi, ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\phi^2$). Since

$$(\xi^1)^2 - (\xi^0)^2 = \rho^2 \qquad , \qquad \frac{\xi^0}{\xi^1} = \tanh\eta \ , \tag{1.25}$$

by construction the lines of constant ρ, $\rho = \rho_0$, are hyperbolas, $(\xi^1)^2 - (\xi^0)^2 = \rho_0^2$, while the lines of constant $\eta = \eta_0$ are straight lines through the origin, $\xi^0 = (\tanh\eta_0)\xi^1$. The null lines $\xi^0 = \pm\xi^1$ correspond to $\eta = \pm\infty$. Thus the Rindler coordinates cover the first quadrant $\xi^1 > |\xi^0|$ of Minkowski space and can be used as coordinates there.

Along the worldline of an observer with constant ρ one has $d\tau = \rho_0 d\eta$, so that his proper time parametrised path is

$$\xi^0(\tau) = \rho_0 \sinh\tau/\rho_0 \qquad \xi^1(\tau) = \rho_0\cosh\tau/\rho_0 \ , \tag{1.26}$$

and his 4-velocity is given by

$$u^0 = \frac{d}{d\tau}\xi^0(\tau) = \cosh\tau/\rho_0 \qquad u^1 = \frac{d}{d\tau}\xi^1(\tau) = \sinh\tau/\rho_0 \ . \tag{1.27}$$

These satisfy $-(u^0)^2 + (u^1)^2 = -1$ (as they should), and comparison with (1.20,1.21) shows that the observer's (constant) acceleration is $\mathsf{a} = 1/\rho_0$.

Even though (1.24) is just the metric of Minkwoski space-time, written in accelerated coordinates, this metric exhibits a number of interesting features that are prototypical of more general metrics that one encounters in general relativity:

1. First of all, we notice that the coefficients of the line element (metric) in (1.24) are no longer constant (space-time independent). Since in the case of constant acceleration we are just describing a "fake" gravitational field, this dependence on the coordinates is such that it can be completely and globally eliminated by passing to appropriate new coordinates (namely inertial Minkowski coordinates). Since, by the equivalence principle, locally an observer cannot distinguish between a fake and a "true" gravitational field, this now suggests that a "true" gravitational field can be described in terms of a space-time coordinate dependent line-element $ds^2 = g_{\alpha\beta}(x)dx^\alpha dx^\beta$ where the coordinate dependence on the x^α is now such that it cannot be eliminated globally by a suitable choice of coordinates.

2. We observe that (1.24) appears to be ill-defined at $\rho = 0$. However, in this case we already know that this is a mere *coordinate singularity* at $\rho = 0$ (akin to the coordinate singularity at the origin of standard polar coordinates in the Cartesian plane). More generally, whenever a metric written in some coordinate system appears to exhibit some singular behaviour, one needs to investigate whether this is just a coordinate singularity or a true singularity of the gravitational field itself.

3. The above coordinates do not just fail at $\rho = 0$, they actually fail to cover large parts of Minkowski space. Thus the next lesson is that, given a metric in some coordinate system, one has to investigate if the space-time described in this way needs to be extended beyond the range of the original coordinates. One way to analyse this question (which we will make extensive use of in section 13 when trying to understand and come to terms with black holes) is to study light rays or the worldlines of freely falling (inertial) observers. In the present example, it is evident that a stationary inertial observer (at fixed value of ξ^1, say, with $\xi^0 = \tau$ his proper time), will "discover" that $\eta = +\infty$ is not the end of the world (he crosses this line at finite proper time $\tau = \xi^1$) and that Minkowski space continues (at the very least) into the quadrant $\xi^0 > |\xi^1|$.

4. Related to this is the behaviour of lightcones when expressed in terms of the coordinates (η, ρ) or when drawn in the (η, ρ)-plane (do this!). These lightcones satisfy $ds^2 = 0$, i.e.

$$\rho^2 d\eta^2 = d\rho^2 \quad \Rightarrow \quad d\eta = \pm\rho^{-1} d\rho \ . \tag{1.28}$$

describing outgoing (ρ grows with η) respectively ingoing (ρ decreases with increasing η) light rays. These lightcones have the familiar Minkowskian shape at $\rho = 1$, but the lightcones open up for $\rho > 1$ and become more and more narrow for $\rho \to 0$, once again exactly as we will find for the Schwarzschild black hole metric (see Figure 14 in section 13.6).

5. Finally we note that there is a large region of Minkowski space that is "invisible" to the constantly accelerated observers. While a static observer will eventually receive information from any event anywhere in space-time (his past lightcone will eventually cover all of Minkowski space ...), the past lightcone of one of the Rindler accelerated observers (whose worldlines asymptote to the lightcone direction $\xi^0 = \xi^1$) will asymptotically only cover one half of Minkowski space, namely the region $\xi^0 < \xi^1$. Thus any event above the line $\xi^0 = \xi^1$ will forever be invisible to this class of observers. Such an observer-dependent horizon has some similarities with the *event horizon* characterising a black hole (section 13).

We now consider the effect of arbitrary (general) coordinate transformations on the laws of special relativity and the geometry of Minkowski space(-time). Let us see what the equation of motion (1.13) of a free massive particle looks like when written in some other (non-inertial, accelerating) coordinate system. It is extremely useful for bookkeeping purposes and for avoiding algebraic errors to use different kinds of indices for different coordinate systems. Thus we will call the new coordinates $x^\mu(\xi^B)$ and not, say, $x^A(\xi^B)$.

First of all, proper time should not depend on which coordinates we use to describe the motion of the particle (the particle couldn't care less what coordinates we experimenters or observers use). [By the way: this is the best way to resolve the so-called 'twin-paradox': It doesn't matter which reference system you use - the accelerating twin in the rocket will always be younger than her brother when they meet again.] Thus

$$\begin{aligned} d\tau^2 &= -\eta_{AB} d\xi^A d\xi^B \\ &= -\eta_{AB} \frac{\partial \xi^A}{\partial x^\mu} \frac{\partial \xi^B}{\partial x^\nu} dx^\mu dx^\nu \quad . \end{aligned} \tag{1.29}$$

Here

$$J^A_\mu(x) = \frac{\partial \xi^A}{\partial x^\mu} \tag{1.30}$$

is the Jacobi matrix associated to the coordinate transformation $\xi^A = \xi^A(x^\mu)$, and we will make the assumnption that (locally) this matrix is non-degenerate, thus has an inverse $J^\mu_A(x)$ or $J^\mu_A(\xi)$ which is the Jacobi matrix associated to the inverse coordinate transformation $x^\mu = x^\mu(\xi^A)$,

$$J^A_\mu J^\mu_B = \delta^A_B \qquad J^\mu_A J^A_\nu = \delta^\mu_\nu \quad . \tag{1.31}$$

We see that in the new coordinates, proper time and distance are no longer measured by the Minkowski metric, but by

$$d\tau^2 = -g_{\mu\nu}(x) dx^\mu dx^\nu \quad , \tag{1.32}$$

where the *metric tensor* (or *metric* for short) $g_{\mu\nu}(x)$ is

$$g_{\mu\nu}(x) = \eta_{AB} \frac{\partial \xi^A}{\partial x^\mu} \frac{\partial \xi^B}{\partial x^\nu} \quad . \tag{1.33}$$

The fact that the Minkowski metric written in the coordinates x^μ in general depends on x should not come as a surprise - after all, this also happens when one writes the Euclidean metric in spherical coordinates etc.

It is easy to check, using (1.31), that the inverse metric, which we will denote by $g^{\mu\nu}$,

$$g^{\mu\nu}(x) g_{\nu\lambda}(x) = \delta^\mu_\lambda \quad , \tag{1.34}$$

is given by

$$g^{\mu\nu}(x) = \eta^{AB} \frac{\partial x^\mu}{\partial \xi^A} \frac{\partial x^\nu}{\partial \xi^B} \quad . \tag{1.35}$$

We will have much more to say about the metric below and, indeed, throughout this course.

Turning now to the equation of motion, the usual rules for a change of variables give

$$\frac{d}{d\tau}\xi^A = \frac{\partial \xi^A}{\partial x^\mu}\frac{dx^\mu}{d\tau} \quad , \tag{1.36}$$

where $\frac{\partial \xi^A}{\partial x^\mu}$ is an invertible matrix at every point. Differentiating once more, one finds

$$\begin{aligned} \frac{d^2}{d\tau^2}\xi^A &= \frac{\partial \xi^A}{\partial x^\mu}\frac{d^2 x^\mu}{d\tau^2} + \frac{\partial^2 \xi^A}{\partial x^\nu \partial x^\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} \\ &= \frac{\partial \xi^A}{\partial x^\mu}\left[\frac{d^2 x^\mu}{d\tau^2} + \frac{\partial x^\mu}{\partial \xi^B}\frac{\partial^2 \xi^B}{\partial x^\nu \partial x^\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}\right] \quad . \end{aligned} \tag{1.37}$$

Thus, since the matrix appearing outside the square bracket is invertible, in terms of the coordinates x^μ the equation of motion, or the equation for a straight line in Minkowski space, becomes

$$\frac{d^2 x^\mu}{d\tau^2} + \frac{\partial x^\mu}{\partial \xi^A}\frac{\partial^2 \xi^A}{\partial x^\nu \partial x^\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \quad . \tag{1.38}$$

The second term in this equation, which we will write as

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\ \nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \quad , \tag{1.39}$$

where

$$\Gamma^\mu_{\ \nu\lambda} = \frac{\partial x^\mu}{\partial \xi^A}\frac{\partial^2 \xi^A}{\partial x^\nu \partial x^\lambda} \quad , \tag{1.40}$$

represents a pseudo-force or fictitious gravitational force (like a centrifugal force or the Coriolis force) that arises whenever one describes inertial motion in non-inertial coordinates. This term is absent for linear coordinate transformations $\xi^A(x^\mu) = M^A_\mu x^\mu$. In particular, this means that the equation (1.13) is invariant under Lorentz transformations, as it should be.

While (1.39) looks a bit complicated, it has one fundamental and attractive feature which will also make it the prototype of the kind of equations that we will be looking for in general. This feature is its *covariance* under general coordinate transformations, which means that the equation takes the same form in any coordinate system. Indeed, this covariance is in some sense tautologically true since the coordinate system $\{x^\mu\}$ that we have chosen is indeed arbitrary. However, it is instructive to see how this comes about by explicitly transforming (1.39) from one coordinate system to another.

Thus consider transforming (1.13) to another coordinate system $\{y^{\mu'}\}$. Following the same steps as above, one thus arrives at the y-version of (1.37), namely

$$\frac{d^2}{d\tau^2}\xi^A = \frac{\partial \xi^A}{\partial y^{\mu'}}\left[\frac{d^2 y^{\mu'}}{d\tau^2} + \frac{\partial y^{\mu'}}{\partial \xi^B}\frac{\partial^2 \xi^B}{\partial y^{\nu'} \partial y^{\lambda'}}\frac{dy^{\nu'}}{d\tau}\frac{dy^{\lambda'}}{d\tau}\right] \quad . \tag{1.41}$$

Equating this result to (1.37) and using

$$\frac{\partial y^{\mu'}}{\partial x^{\mu}} = \frac{\partial y^{\mu'}}{\partial \xi^A}\frac{\partial \xi^A}{\partial x^{\mu}} \ , \tag{1.42}$$

one finds

$$\frac{d^2 y^{\mu'}}{d\tau^2} + \Gamma^{\mu'}_{\ \nu'\lambda'}\frac{dy^{\nu'}}{d\tau}\frac{dy^{\lambda'}}{d\tau} = \frac{\partial y^{\mu'}}{\partial x^{\mu}}\left[\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\ \nu\lambda}\frac{dx^{\nu}}{d\tau}\frac{dx^{\lambda}}{d\tau}\right] \tag{1.43}$$

Thus the geodesic equation transforms in the simplest possible non-trival way under coordinate transformations $x \to y$, namely with the Jacobian matrix $\partial(y)/\partial(x)$. We will see later that this transformation behaviour characterises/defines tensors, in this particular case a vector (or contravriant tensor of rank 1).

In particular, since this matrix is assumed to be invertible, we reach the conclusion that the left hand side of (1.43) is zero if the term in square brackets on the right hand side is zero,

$$\frac{d^2 y^{\mu'}}{d\tau^2} + \Gamma^{\mu'}_{\ \nu'\lambda'}\frac{dy^{\nu'}}{d\tau}\frac{dy^{\lambda'}}{d\tau} = 0 \quad \Leftrightarrow \quad \frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\ \nu\lambda}\frac{dx^{\nu}}{d\tau}\frac{dx^{\lambda}}{d\tau} = 0 \tag{1.44}$$

This is what is meant by the statement that the equation takes the same form in any coordinate system. We see that in this case this is achieved by having the equation transform in a particularly simple way under coordinate transformations, namely as a tensor.

Remarks:

1. We will see below that, in general, that is for an arbitrary metric, not necessarily related to the Minkowski metric by a coordinate transformation, the equation for a *geodesic*, i.e. a path that extremises proper time or proper distance, takes the form (1.39), where the (pseudo-)force terms can be expressed in terms of the first derivatives of the metric as

$$\begin{aligned} \Gamma^{\mu}_{\ \nu\lambda} &= g^{\mu\rho}\Gamma_{\rho\nu\lambda} \\ \Gamma_{\rho\nu\lambda} &= \tfrac{1}{2}(g_{\rho\nu},_{\lambda} + g_{\rho\lambda},_{\nu} - g_{\nu\lambda},_{\rho}) \ . \end{aligned} \tag{1.45}$$

   [You should check yourself that plugging the metric (1.33) into this equation, you find the result (1.40).]

2. In this sense, the metric plays the role of a potential for the pseudo-force and, more generally, for the gravitational force, and will thus come to play the role of the fundamental dynamical variable of gravity. Also, in this more general context the $\Gamma^{\mu}_{\ \nu\lambda}$ are referred to as the *Christoffel symbols* of the metric or (in more fancy terminology) the components of the Levi-Civita connection, a privileged connection on the tangent (or frame) bundle of a manifold equipped with a metric.

1.4 Metrics and Coordinate Transformations

Above we saw that the motion of free particles in Minkowski space in curvilinear coordinates is described in terms of a modified metric, $g_{\mu\nu}$, and a force term $\Gamma^{\mu}_{\ \nu\lambda}$ representing the 'pseudo-force' on the particle. Thus the Einstein Equivalence Principle suggests that an appropriate description of true gravitational fields is in terms of a metric tensor $g_{\mu\nu}(x)$ (and its associated Christoffel symbols) which can only locally be related to the Minkowski metric via a suitable coordinate transformation (to locally inertial coordinates). Thus our starting point will now be a space-time equipped with some metric $g_{\mu\nu}(x)$, which we will assume to be symmetric and non-degenerate, i.e.

$$g_{\mu\nu}(x) = g_{\nu\mu}(x) \qquad \mathsf{det}(g_{\mu\nu}(x)) \neq 0 \ . \tag{1.46}$$

A space-time equipped with a metric tensor $g_{\mu\nu}(x)$ is called a metric space-time or (pseudo-)Riemannian space-time. Here "Riemannian" usually refers to a space equipped with a positive-definite metric (all eigenvalues positive), while pseudo-Riemannian (or Lorentzian) refers to a space-time with a metric with one negative and 3 (or 27, or whatever) positive eigenvalues. The metric encodes the information how to measure (spatial and temporal) distances, as well as areas, volumes etc., via the associated line element

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \ . \tag{1.47}$$

<u>Remarks:</u>

1. Examples that you may be familar with are the metrics on the 2-sphere or 3-sphere of radius R in spherical coordinates,

$$\begin{aligned} ds^2(S^2) &= R^2(d\theta^2 + \sin^2\theta d\phi^2) \\ ds^2(S^3) &= R^2(d\alpha^2 + \sin^2\alpha(d\theta^2 + \sin^2\theta d\phi^2)) \ . \end{aligned} \tag{1.48}$$

   These metrics can of course be elevated to space-time metrics by adding e.g. a $(-dt^2)$, and for example a space-time metric describing a spatially spherical universe with a time-dependent radius (expansion of the universe!) might be described by the line element

$$ds^2 = -dt^2 + a(t)^2\left(d\alpha^2 + \sin^2\alpha(d\theta^2 + \sin^2\theta d\phi^2)\right) \ . \tag{1.49}$$

2. A metric determines a geometry, but different metrics may well determine the same geometry, namely those metrics which are just related by coordinate transformations. In particular, distances should not depend on which coordinate system is used. Hence, changing coordinates from the $\{x^\mu\}$ to new coordinates $\{y^{\mu'}(x^\mu)\}$ and demanding that

$$g_{\mu\nu}(x)dx^\mu dx^\nu = g_{\mu'\nu'}(y)dy^{\mu'}dy^{\nu'} \ , \tag{1.50}$$

one finds that under a coordinate transformation the metric transforms as

$$g_{\mu'\nu'} = g_{\mu\nu}\frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial x^\nu}{\partial y^{\nu'}} \equiv J^\mu_{\mu'}J^\nu_{\nu'}\,g_{\mu\nu} \quad . \tag{1.51}$$

3. Objects which transform in such a nice and simple way under coordinate transformations are known as *tensors* - the metric is an example of what is known as (and we will get to know as) a covariant symmetric rank two tensor. We will study tensors in much more detail and generality later, starting in section 3.

4. One point to note about this transformation behaviour is that if in one coordinate system the metric tensor has one negative and three positive eigenvalues (as in a locally inertial coordinate system), then the same will be true in any other coordinate system (even though the eigenvalues themselves will in general be different) - this statement should be familiar from linear algebra as Sylvester's law of inertia.

5. In a pseudo-Riemannian space-time one has the same distinction between spacelike, timelike and lightlike separations as in Minkowski space(-time). Spacelike distances correspond to $ds^2 > 0$, timelike distances to $d\tau^2 = -ds^2 > 0$, and null or lightlike distances to $ds^2 = d\tau^2 = 0$. In particular, a vector $V^\mu(x)$ at a point x is called spacelike if $g_{\mu\nu}(x)V^\mu(x)V^\nu(x) > 0$ etc., and a curve $x^\mu(\lambda)$ is called spacelike if its tangent vector is everywhere spacelike etc. Using the definition of a vector in general relativity (to be introduced in section 3), namely an object that transforms in the obvious way, with the Jacobi matrix, under coordinate transformations, one sees that $g_{\mu\nu}(x)V^\mu(x)V^\nu(x)$ is a scalar, i.e. invariant under coordinate transformation, and hence the statement that a vector is, say, spacelike is a coordinate-independent statement, as it should be.

6. By drawing the coordinate grid determined by the metric tensor, one can convince onseself that in general a metric space or space-time need not or cannot be flat. Example: the coordinate grid of the metric $d\theta^2 + \sin^2\theta d\phi^2$ cannot be drawn in flat space but can be drawn on the surface of a two-sphere because the infinitesimal parallelograms described by ds^2 degenerate to triangles not just at $\theta = 0$ (as would also be the case for the flat metric $ds^2 = dr^2 + r^2 d\phi^2$ in polar coordinates at $r = 0$), but also at $\theta = \pi$.

At this point the question naturally arises how one can tell whether a given (perhaps complicated looking) metric is just the flat metric written in other coordinates or whether it describes a genuinely curved space-time. We will see later that there is an object, the *Riemann curvature tensor*, constructed from the second derivatives of the metric, which has the property that all of its components vanish if and only if the metric is a coordinate transform of the flat space Minkowski metric. Thus, given a metric, by calculating its curvature tensor one can decide if the metric is just the flat metric in disguise or not. The curvature tensor will be introduced in section 7, and the above statement will be established in section 8.2.

We have seen that the equation for a straight line in Minkowski space, written in arbitrary coordinates, is

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \ , \tag{1.52}$$

where the pseudo-force term $\Gamma^\mu_{\nu\lambda}$ is given by (1.40). We have also seen in (1.45) (provided you checked this) that $\Gamma^\mu_{\nu\lambda}$ can be expressed in terms of the metric (1.33) as

$$\Gamma^\mu_{\nu\lambda} = \tfrac{1}{2}g^{\mu\rho}(g_{\rho\nu},_\lambda + g_{\rho\lambda},_\nu - g_{\nu\lambda},_\rho) \ . \tag{1.53}$$

This gravitational force term is fictitious since it can globally be transformed away by going to the global inertial coordinates ξ^A. The equivalence principle suggests, however, that in general the equation for the worldline of a massive particle, i.e. a path that extremises proper time, in a true gravitational field is also of the above form.

We will now confirm this by deriving the equations for a path that extremises proper time from a variational principle. Recall from special relativity, that the Lorentz-invariant action of a free massive particle with mass m is is

$$S_0 = -m\int d\tau \ , \tag{1.54}$$

with $d\tau^2 = -\eta_{AB}d\xi^A d\xi^B$. We can adopt the same action in the present setting, with the coordinate-invariant Lagrangian

$$d\tau^2 = -g_{\mu\nu}(x)dx^\mu dx^\nu \ . \tag{1.55}$$

Of course m drops out of the variational equations (as it should by the equivalence principle) and we will therefore ignore m in the following. We can also consider spacelike paths that extremise (minimise) proper distance, by using the action

$$S_0 \sim \int ds \tag{1.56}$$

where

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \ . \tag{1.57}$$

One should also consider massless particles, whose worldlines will be null (or lightlike) paths. However, in that case one can evidently not use proper time or proper distance, since these are by definition zero along a null path, $d\tau^2 = 0$. We will come back to this special case, and a unified description of the massive and massless case, below (section 2.1). In all cases, we will refer to the resulting paths as *geodesics*.

In order to perform the variation, it is useful to introduce an arbitrary auxiliary parameter λ in the initial stages of the calculation via

$$d\tau = (-g_{\mu\nu}\tfrac{dx^\mu}{d\lambda}\tfrac{dx^\nu}{d\lambda})^{1/2}d\lambda \ , \tag{1.58}$$

and to write

$$\int d\tau = \int (d\tau/d\lambda)d\lambda = \int (-g_{\mu\nu}\tfrac{dx^\mu}{d\lambda}\tfrac{dx^\nu}{d\lambda})^{1/2}d\lambda \ . \tag{1.59}$$

We are varying the paths

$$x^\mu(\tau) \to x^\mu(\tau) + \delta x^\mu(\tau) \tag{1.60}$$

keeping the end-points fixed, and will denote the τ-derivatives by $\dot{x}^\mu(\tau)$. By the standard variational procedure one then finds

$$\begin{aligned} \delta \int d\tau &= \tfrac{1}{2}\int (-g_{\mu\nu}\tfrac{dx^\mu}{d\lambda}\tfrac{dx^\nu}{d\lambda})^{-1/2}d\lambda \left[-\delta g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} - 2g_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right] \\ &= \frac{1}{2}\int d\tau \left[-g_{\mu\nu},_\lambda \dot{x}^\mu\dot{x}^\nu\delta x^\lambda + 2g_{\mu\nu}\ddot{x}^\nu\delta x^\mu + 2g_{\mu\nu},_\lambda \dot{x}^\lambda\dot{x}^\nu\delta x^\mu\right] \\ &= \int d\tau \left[g_{\mu\nu}\ddot{x}^\nu + \tfrac{1}{2}(g_{\mu\nu},_\lambda + g_{\mu\lambda},_\nu - g_{\nu\lambda},_\mu)\dot{x}^\nu\dot{x}^\lambda\right]\delta x^\mu \end{aligned} \tag{1.61}$$

Here the factor of 2 in the first equality is a consequence of the symmetry of the metric, the second equality follows from an integration by parts, the third from relabelling the indices in one term and using the symmetry in the indices of $\dot{x}^\lambda\dot{x}^\nu$ in the other.

If we now adopt the definition (1.53) for an arbitrary metric,

$$\Gamma^\mu_{\ \nu\lambda} = \tfrac{1}{2}g^{\mu\rho}(g_{\rho\nu},_\lambda + g_{\rho\lambda},_\nu - g_{\nu\lambda},_\rho) \ , \tag{1.62}$$

we can write the result as

$$\delta \int d\tau = \int d\tau g_{\mu\nu}(\ddot{x}^\nu + \Gamma^\nu_{\ \rho\lambda}\dot{x}^\rho\dot{x}^\lambda)\delta x^\mu \ . \tag{1.63}$$

Thus we see that indeed the equations of motion for a massive particle in an arbitrary gravitational field are

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\ \nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \ . \tag{1.64}$$

We will have much more to say about geodesics and variational principles in section 2.

1.6 Christoffel Symbols and Coordinate Transformations

The Christoffel symbols play the role of the gravitational force term, and thus in this sense the components of the metric play the role of the gravitational potential. These Christoffel symbols play an important role not just in the geodesic equation but, as we will see later on, more generally in the definition of a covariant derivative operator and the construction of the curvature tensor.

Two elementary important properties of the Christoffel symbols are that they are symmetric in the second and third indices,

$$\Gamma_{\mu\nu\lambda} = \Gamma_{\mu\lambda\nu} \quad , \quad \Gamma^\mu_{\ \nu\lambda} = \Gamma^\mu_{\ \lambda\nu} \ , \tag{1.65}$$

and that symmetrising $\Gamma_{\mu\nu\lambda}$ over the first pair of indices one finds

$$\Gamma_{\mu\nu\lambda} + \Gamma_{\nu\mu\lambda} = g_{\mu\nu,\lambda} \quad . \tag{1.66}$$

Knowing how the metric transforms under coordinate transformations, we can now also determine how the Christoffel symbols (1.53) and the geodesic equation transform. A straightforward but not particularly inspiring calculation (which you should nevertheless do) gives

$$\Gamma^{\mu'}_{\ \nu'\lambda'} = \Gamma^{\mu}_{\ \nu\lambda} \frac{\partial y^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial y^{\nu'}} \frac{\partial x^{\lambda}}{\partial y^{\lambda'}} + \frac{\partial y^{\mu'}}{\partial x^{\mu}} \frac{\partial^2 x^{\mu}}{\partial y^{\nu'} \partial y^{\lambda'}} \quad . \tag{1.67}$$

Thus, $\Gamma^{\mu}_{\ \nu\lambda}$ transforms inhomogenously under coordinate transformations. If only the first term on the right hand side were present, then $\Gamma^{\mu}_{\ \nu\lambda}$ would be a tensor. However, the second term is there precisely to compensate for the fact that $\ddot{x}^{\mu}$ is also not a tensor - the combined geodesic equation transforms in a nice way under coordinate transformations. Namely, after another not terribly inspiring calculation (which you should nevertheless also do at least once in your life) , one finds

$$\frac{d^2 y^{\mu'}}{d\tau^2} + \Gamma^{\mu'}_{\ \nu'\lambda'} \frac{dy^{\nu'}}{d\tau} \frac{dy^{\lambda'}}{d\tau} = \frac{\partial y^{\mu'}}{\partial x^{\mu}} \left[\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\ \nu\lambda} \frac{dx^{\nu}}{d\tau} \frac{dx^{\lambda}}{d\tau} \right] \quad . \tag{1.68}$$

This is analogous to the result (1.43) that we had obtained before in Minkoswki space, and the same remarks about covariance and tensors etc. apply.

Remarks:

1. That the geodesic equation transforms in this simple way (namely as a vector) should not come as a surprise. We obtained this equation as a variational equation. The Lagrangian itself is a scalar (invariant under coordinate transformations), and the variation δx^{μ} is (i.e. transforms like) a vector. Putting these pieces together, one finds the desired result. [This comment may become less mysterious after the discussion of tensors and Lie derivatives ...]

2. There is of course a very good physical reason for why the force term in the geodesic equation (which, incidentally, is quadratic in the velocities, quite peculiar) is not tensorial. This simply reflects the equivalence principle that locally, at a point (or in a sufficiently small neighbourhood of a point) you can eliminate the gravitational force by going to a freely falling (inertial) coordinate system. This would not be possible if the gravitational force term in the equation of motion for a particle were tensorial.

2 The Physics and Geometry of Geodesics

2.1 An Alternative Variational Principle for Geodesics

As we have already noted, there is a problem with the above action principle for massless particles (null geodesics). For this reason and many other practical purposes (the square root in the action is awkward) it is much more convenient to use, instead of the action

$$S_0[x] = -m\int d\tau = -m\int d\lambda\sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}} \equiv \int d\lambda\, \mathcal{L}_0^\lambda \tag{2.1}$$

the simpler Lagrangian

$$\mathcal{L} = \tfrac{1}{2}g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{2.2}$$

and action

$$S_1[x] = \int d\lambda\, \mathcal{L} \;\; . \tag{2.3}$$

Before discussing and establishing the relation between these two actions, let us first verify that S_1 really leads to the same equations of motion as S_0. Either by direct variation of the action, or by using the Euler-Lagrange equations

$$\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} - \frac{\partial\mathcal{L}}{\partial x^\mu} = 0 \;\; , \tag{2.4}$$

one finds that the action is indeed extremised by the solutions to the equation

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{d\lambda}\frac{dx^\lambda}{d\lambda} = 0 \;\; . \tag{2.5}$$

This is identical to the geodesic equation derived from S_0 (with $\lambda \to \tau$, the proper time).

Now let us turn to the relationship between, and comparison of, the two actions S_0 and S_1. The first thing to notice is that S_0 is manifestly parametrisation-invariant, i.e. independent of how one parametrises the path. The reason for this is that $d\tau = (d\tau/d\lambda)d\lambda$ is evidently independent of λ. This is not the case for S_1, which changes under parametrisations or, put more positively, singles out a preferred parametrisation (or, more precicesly, class of parametrisations). We will discuss this in somewhat more detail in section 2.2.

Thus, what is the relation (if any) between the two actions? In order to explain this, it will be useful to introduce an additional field $e(\lambda)$ (i.e. in addition to the $x^\alpha(\lambda)$), and a "master action" (or parent action) S which we can relate to both S_0 and S_1. Consider the action

$$S[x,e] = \tfrac{1}{2}\int d\lambda\left(e(\lambda)^{-1}g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} - m^2e(\lambda)\right) = \int d\lambda\left(e(\lambda)^{-1}\mathcal{L} - \tfrac{1}{2}m^2e(\lambda)\right) \tag{2.6}$$

The crucial property of this action is that it is parametrisation invariant provided that one declares $e(\lambda)$ to transform appropriately. It is easy to see that under a transformation $\lambda \to \bar{\lambda} = f(\lambda)$, with

$$\bar{x}^\alpha(\bar{\lambda}) = x^\alpha(\lambda) \qquad d\bar{\lambda} = f'(\lambda)d\lambda \tag{2.7}$$

the action S is invariant provided that $e(\lambda)$ transforms such that $e(\lambda)d\lambda$ is invariant, i.e.

$$\bar{e}(\bar{\lambda})d\bar{\lambda} \stackrel{!}{=} e(\lambda)d\lambda \quad \Rightarrow \quad \bar{e}(\bar{\lambda}) = e(\lambda)/f'(\lambda) \ . \tag{2.8}$$

The first thing to note now is that, courtesy of this parametrisation invariance, we can always choose a "gauge" in which $e(\lambda) = 1$. With this choice, the action $S[x,e]$ manifestly reduces to the action $S_1[x]$ modulo an irrelevant field-independent constant,

$$S[x, e = 1] = \int d\lambda\, \mathcal{L} - \tfrac{1}{2}m^2 \int d\lambda = S_1[x] + \text{const.} \ . \tag{2.9}$$

Thus we can regard S_1 as a gauge-fixed version of S (no wonder it is not parametrisation invariant ...). We will come back to the small residual gauge invariance (reparametrisations that preserve the gauge condition $e(\lambda) = 1$) below.

Alternatively, instead of fixing the gauge, we can try to eliminate $e(\lambda)$ (which appears purely algebraically, i.e. without derivatives, in the action) by its equation or motion. Varying $S[x,e]$ with respect to $e(\lambda)$, one finds the constraint

$$g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} + m^2 e(\lambda)^2 = 0 \ . \tag{2.10}$$

This is just the usual mass-shell condition in disguise. It suggests that a better gauge fixing than $e(\lambda) = 1$ would have been $e(\lambda) = m^{-1}$. However, the sole effect of this would have been to replace $\mathcal{L}$ in (2.9) by $m\mathcal{L}$,

$$e(\lambda) = 1 \to e(\lambda) = m^{-1} \quad \Rightarrow \quad \mathcal{L} \to m\mathcal{L} \ . \tag{2.11}$$

In any caser, for a massive particle, $m^2 \neq 0$, one can also solve (2.10) for $e(\lambda)$,

$$e(\lambda) = m^{-1}\sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}} \ . \tag{2.12}$$

Using this to eliminate $e(\lambda)$ from the action, one finds

$$S[x, e = m^{-1}\sqrt{\ldots}] = -m\int d\lambda \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}} = -m\int d\tau = S_0[x] \ . \tag{2.13}$$

Thus for $m^2 \neq 0$ we find exactly the original action (integral of the proper time) $S_0[x]$ (and since we haven't touched or fixed the parametrisation invariance, no wonder that S_0 is parametrisation invariant). Thus we have elucidated the common origin of S_0 and S_1 for a massive particle.

<u>Remarks:</u>

1. An added benefit of the master action $S[x,e]$ is that it also makes perfect sense for a massless particle. For $m^2 = 0$, the mass shell condition

$$g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{2.14}$$

says that these particles move along null lines, and the action reduces to

$$S[x,e] = \tfrac{1}{2}\int d\lambda\; e(\lambda)^{-1} g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} \tag{2.15}$$

which is parametrisation invariant but can (as in the massive case) be fixed to $e(\lambda) = 1$, upon which the action reduces to $S_1[x]$. Thus we see that $S_1[x]$ indeed provides a simple and unfied action for both massive and massless particles, and in both cases the resulting equation of motion is the (affinely parametrised) geodesic equation (2.5),

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0 \ . \tag{2.16}$$

2. One important consequence of (2.16) is that the quantity $\mathcal{L}$ is a constant of motion, i.e. constant along the geodesic,

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0 \quad\Rightarrow\quad \frac{d}{d\lambda}\left(g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}\right) = 0 \ . \tag{2.17}$$

This useful result can be understood and derived in a variety of ways, the least insightful of which is direct calculation. Nevertheless, this is straightforward and a good exercise in Γ-ology. An alternative derivation will be given in section 4, using the concept of 'covariant derivative along a curve'.

From the present (action-based) perspective it is most useful to think of this as the conserved quantity associated (via Noether's theorem) to the invariance of the action $S_1[x]$ under translations in λ. Note that evidently $S_1[x]$ has this invariance (as there is no explicit dependence on λ) and that this invariance is precisely the residual parametrisation invariance $f(\lambda) = \lambda + a$, $f'(\lambda) = 1$, that leaves invariant the "gauge" condition $e(\lambda) = 1$. For an infinitesimal constant λ-translation one has $\delta x^\alpha(\lambda) = dx^\alpha/d\lambda$ etc., so that

$$\frac{\partial}{\partial\lambda}\mathcal{L} = 0 \quad\Rightarrow\quad \delta\mathcal{L} = \frac{d}{d\lambda}\mathcal{L} = \frac{d}{d\lambda}\left(\frac{\partial\mathcal{L}}{\partial(dx^\alpha/d\lambda)}\frac{dx^\alpha}{d\lambda}\right) + \quad \text{Euler} - \text{Lagrange} \ . \tag{2.18}$$

Thus via Noether's theorem the associated conserved charge for a solution to the Euler-Lagrange equations is the Legendre transform

$$\mathcal{H} = \left(\frac{\partial\mathcal{L}}{\partial(dx^\alpha/d\lambda)}\frac{dx^\alpha}{d\lambda}\right) - \mathcal{L} \tag{2.19}$$

of the Lagrangian (aka the Hamiltonian, once expressed in terms of the momenta). In the case at hand, with the Lagrangian $\mathcal{L}$ consisting of a purely quadratic term in the velocities (the $dx^\alpha/d\lambda$), the Hamiltonian is equal to the Lagrangian, and hence the Lagrangian $\mathcal{L}$ itself is conserved,

$$\mathcal{H} = \mathcal{L} \quad , \quad \frac{d}{d\lambda}\mathcal{L}|_{\text{solution}} = 0 \ . \tag{2.20}$$

3. This is as it should be: something that starts off as a massless particle will remain a massless particle etc. If one imposes the initial condition

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\bigg|_{\lambda=0} = \epsilon \ , \tag{2.21}$$

then this condition will be satisfied for all λ. In particular, therefore, one can choose $\epsilon = \mp 1$ for timelike (spacelike) geodesics, and λ can then be identified with proper time (proper distance), while the choice $\epsilon = 0$ sets the initial conditions appropriate to massless particles (for which λ is then not related to proper time or proper distance).

2.2 Affine and Non-affine Parametrisations

To understand the significance of how one parametrises the geodesic, observe that the geodesic equation itself,

$$\ddot{x}^\mu + \Gamma^\mu_{\ \nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0 \ , \tag{2.22}$$

is not parametrisation invariant. Indeed, consider a change of parametrisation $\tau \to \sigma = f(\tau)$. Then

$$\frac{dx^\mu}{d\tau} = \frac{df}{d\tau}\frac{dx^\mu}{d\sigma} \ , \tag{2.23}$$

and therefore the geodesic equation written in terms of σ reads

$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\ \nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma} = -\frac{\ddot{f}}{\dot{f}^2}\frac{dx^\mu}{d\sigma} \ . \tag{2.24}$$

Thus the geodesic equation retains its form only under affine changes of the proper time parameter τ, $f(\tau) = a\tau + b$, and parameters $\sigma = f(\tau)$ related to τ by such an affine transformation are known as affine parameters.

From the first variational principle, based on S_0, the term on the right hand side arises in the calculation of (1.61) from the integration by parts if one does not switch back from λ to the affine parameter τ. The second variational principle, based on S_1 and the Lagrangian $\mathcal{L}$, on the other hand, always and automatically yields the geodesic equation in affine form.

Conversely, if we find a curve that satisfies

$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\ \nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma} = \kappa(\sigma)\frac{dx^\mu}{d\sigma} \ , \tag{2.25}$$

for some function $\kappa(\sigma)$, we can deduce that this curve is the trajectory of a geodesic, but that it is simply not parametrised by an affine parameter (like proper time in the case of a timelike curve). Comparison of (2.24) and (2.25) shows that, given $\kappa(\sigma)$, an affine parameter τ is determined by

$$\kappa(f(\tau)) = -\frac{\ddot{f}}{\dot{f}^2} \quad \Leftrightarrow \quad \kappa(\sigma) = \frac{d}{d\sigma} \ln \frac{d\tau}{d\sigma} \tag{2.26}$$

or

$$\frac{d\tau}{d\sigma} = \mathrm{e}^{\int^\sigma ds\; \kappa(s)} \quad . \tag{2.27}$$

In the following, whenever we talk about geodesics we will practically always have in mind the variational principle based on S_1 leading to the geodesic equation (2.16) in affinely parametrised form.

However, it should be kept in mind that sometimes non-affine parameters appear naturally. For instance, it is occasionally convenient to parametrise timelike geodesics in a geometry with coordinates $x^\alpha = (x^0 = t, x^k)$ not by $x^\alpha = x^\alpha(\tau)$, where τ is the proper time along the geodesic, but rather as $x^k = x^k(t)$. This is the same curve, but described with respect to coordinate time (which could for instance agree with the proper time of some other, perhaps stationary, observer). The curve $t \to (t, x^k(t))$ will not be an affinely parametrised curve unless t satisfies the geodesic equation $\ddot{t} = 0 \leftrightarrow t = a\tau + b$.

One occasion where this will play a role (and from where I have borrowed the symbol κ for the "inaffinity") is in our discussion, much later, of the horizon of a black hole, where the lack of a certain coordinate to be an affine parameter is directly related to the physical properties of black holes (see section 13.8). In this context κ is known as the surface gravity of a black hole.

2.3 A Simple Example

It is high time to consider an example. We will consider the simplest non-trivial metric, namely the standard Euclidean metric on $\mathbb{R}^2$ in polar coordinates. Thus the line element is

$$ds^2 = dr^2 + r^2 d\phi^2 \tag{2.28}$$

and the non-zero components of the metric are $g_{rr} = 1, g_{\phi\phi} = r^2$. Since this metric is diagonal, the components of the inverse metric $g^{\mu\nu}$ are $g^{rr} = 1$ and $g^{\phi\phi} = r^{-2}$.

A remark on notation: since μ, ν in $g_{\mu\nu}$ are coordinate indices, we should really have called $x^1 = r$, $x^2 = \phi$, and written $g_{11} = 1, g_{22} = r^2$, etc. However, writing g_{rr} etc. is more informative and useful since one then knows that this is the (rr)-component of the metric without having to remember if one called $r = x^1$ or $r = x^2$. In the following we will frequently use this kind of notation when dealing with a specific coordinate system, while we retain the index notation $g_{\mu\nu}$ etc. for general purposes.

The Christoffel symbols of this metric are to be calculated from

$$\Gamma_{\mu\nu\lambda} = \tfrac{1}{2}(g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\nu\lambda,\mu}) \ . \tag{2.29}$$

Since the only non-trivial derivative of the metric is $g_{\phi\phi,r} = 2r$, only Christoffel symbols with exactly two ϕ's and one r are non-zero,

$$\begin{aligned} \Gamma_{r\phi\phi} &= \tfrac{1}{2}(g_{r\phi,\phi} + g_{r\phi,\phi} - g_{\phi\phi,r}) = -r \\ \Gamma_{\phi\phi r} &= \Gamma_{\phi r\phi} = r \ . \end{aligned} \tag{2.30}$$

Thus, since the metric is diagonal, the non-zero $\Gamma^{\mu}{}_{\nu\lambda}$ are

$$\begin{aligned} \Gamma^{r}{}_{\phi\phi} &= g^{r\mu}\Gamma_{\mu\phi\phi} = g^{rr}\Gamma_{r\phi\phi} = -r \\ \Gamma^{\phi}{}_{r\phi} &= \Gamma^{\phi}{}_{\phi r} = g^{\phi\mu}\Gamma_{\mu r\phi} = g^{\phi\phi}\Gamma_{\phi r\phi} = \frac{1}{r} \ . \end{aligned} \tag{2.31}$$

Note that here it was even convenient to use a hybrid notation, as in $g^{r\mu}$, where r is a coordinate and μ is a coordinate index. Once again, it is very convenient to permit oneself to use such a mixed notation.

In any case, having assembled all the Christoffel symbols, we can now write down the geodesic equations (one again in the convenient hybrid notation). For r one has

$$\ddot{r} + \Gamma^{r}{}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0 \ , \tag{2.32}$$

which, since the only non-zero $\Gamma^{r}{}_{\mu\nu}$ is $\Gamma^{r}{}_{\phi\phi}$, reduces to

$$\ddot{r} - r\dot{\phi}^2 = 0 \ . \tag{2.33}$$

Likewise for ϕ one finds

$$\ddot{\phi} + \frac{2}{r}\dot{\phi}\dot{r} = 0 \ . \tag{2.34}$$

Here the factor of 2 arises because both $\Gamma^{\phi}{}_{r\phi}$ and $\Gamma^{\phi}{}_{\phi r} = \Gamma^{\phi}{}_{r\phi}$ contribute.

Now this equation is supposed to describe geodesics in $\mathbb{R}^2$, i.e. straight lines. This can be verified in general (but, in general, polar coordinates are of course not particularly well suited to describe straight lines). However, it is easy to find a special class of solutions to the above equations, namely curves with $\dot{\phi} = \ddot{r} = 0$. These correspond to paths of the form $(r(s), \phi(s)) = (s, \phi_0)$, which are a special case of straight lines, namely straight lines through the origin.

The geodesic equations can of course also be derived as the Euler-Lagrange equations of the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) \ . \tag{2.35}$$

Indeed, one has

$$\begin{aligned} \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{r}} - \frac{\partial\mathcal{L}}{\partial r} &= \ddot{r} - r\dot{\phi}^2 = 0 \\ \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \frac{\partial\mathcal{L}}{\partial\phi} &= r^2\ddot{\phi} + 2r\dot{r}\dot{\phi} = 0 \ , \end{aligned} \tag{2.36}$$

which are obviously identical to the equations derived above.

You may have the impression that getting the geodesic equation in this way, rather than via calculation of the Christoffel symbols first, is much simpler. I agree wholeheartedly. Not only is the Lagrangian approach the method of choice to determine the geodesic equations. It is also frequently the most efficient method to determine the Christoffel symbols. This will be described in the next section.

Another advantage of the Lagrangian formulation is, as in classical mechanics, that it makes it much easier to detect and exploit symmetries. Indeed, you may have already noticed that the above second-order equation for ϕ is overkill. Since the Lagrangian does not depend on ϕ (i.e. it is invariant under rotations), one has

$$\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = 0 \quad , \tag{2.37}$$

which means that $\partial\mathcal{L}/\partial\dot{\phi}$ is a constant of motion, the angular momentum L,

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = r^2\dot{\phi} = L \quad . \tag{2.38}$$

This equation is a first integral of the second-order equation for ϕ. We will come back to this in somewhat more generality below.

The next simplest example to discuss would be the two-sphere with its standard metric $d\theta^2 + \sin^2\theta d\phi^2$. It will appear, in bits and pieces, in the next section to illustrate the general remarks.

2.4 Consequences and Uses of the Euler-Lagrange Equations

Recall from above that the geodesic equation for a metric $g_{\mu\nu}$ can be derived from the Lagrangian $\mathcal{L} = (1/2)g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$

$$\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} - \frac{\partial\mathcal{L}}{\partial x^\mu} = 0 \quad . \tag{2.39}$$

This has several immediate consequences which are useful for the determination of Christoffel symbols and geodesics in practice.

1. Conserved charges / first integrals of the geodesic equation

   Just as in classical mechanics, a coordinate the Lagrangian does not depend on explicitly (a cyclic coordinate) leads to a conserved quantity, assocaited with the translation invariance of the system in that direction. In the present context this means that if, say, $\partial\mathcal{L}/\partial x^1 = 0$ (this means that the coeffcients of the metric do not depend on x^1), then the corresponding momentum

$$p_1 = \partial\mathcal{L}/\partial\dot{x}^1 = g_{1\nu}\dot{x}^\nu \tag{2.40}$$

is conserved along the geodesic.

REMARKS:

(a) One might perhaps have wanted to argue that the definition (and interpretation) of conserved momenta should be based on the physical Lagrangian (2.1)

$$\mathcal{L}_0^\lambda = -m\sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}} \tag{2.41}$$

with action $S = -m\int d\tau$, but this makes no difference since the two momenta are essentially equal: one has

$$\frac{\partial \mathcal{L}_0^\lambda}{\partial(dx^1/d\lambda)} = mp_1 \tag{2.42}$$

with p_1 as defined in (2.40), so that this just supplies us with the additional information that the momenta obtained from the Lagrangian $\mathcal{L}$ should (for a massive particle) be interpreted as momenta per unit mass. This discrepancy could have been avoided by working with the Lagrangian $m\mathcal{L}$ (alternatively: fixing the gauge $e(\lambda) = m^{-1}$ in section 2.1, see (2.11)), but unless or until one starts coupling the particle to fields other than the gravitational field it is unnecessary (and a nuisance) to carry m around all the time.

(b) For example, on the two-sphere the Lagrangian reads

$$\mathcal{L} = \tfrac{1}{2}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \ . \tag{2.43}$$

The angle ϕ is a cyclic variable and the angular momentum (actually angular momentum per unit mass for a massive particle)

$$p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \sin^2\theta\dot{\phi} \tag{2.44}$$

is a conserved quantity. This generalises to conservation of angular momentum for a particle moving in an arbitrary spherically symmetric gravitational field.

(c) Likewise, if the metric is independent of the time coordinate $x^0 = t$, the corresponding conserved quantity

$$p_0 = g_{0\nu}\dot{x}^\nu \equiv -E \tag{2.45}$$

has the interpretation as minus the energy (per unit mass) of the particle, "minus" because, with our sign conventions, $p_0 = -E$ in special relativity. We will discuss the relation between this notion of energy and the notion of energy familiar from special relativity (this requires an asymptotically Minkowski-like metric) in more detail in section 12.1.

(d) We will discuss in more detail in section 2.5 (and then again in section 6) how to detect and describe symmetries and conserved charges in coordinate systems in which the symmetries are not as manifest (via cyclic variables) as above.

2. Reading off (some) geodesics directly from the metric

Another immediate consequence is the following: consider a space or space-time with coordinates $\{y, x^\mu\}$ and a metric of the form $ds^2 = dy^2 + g_{\mu\nu}(x,y)dx^\mu dx^\nu$. Then the coordinate lines of y are geodesics. Indeed, since the Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}(\dot{y}^2 + g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu) \quad , \tag{2.46}$$

the Euler-Lagrange equations are of the form

$$\begin{aligned} \ddot{y} - \tfrac{1}{2}g_{\mu\nu},_y\,\dot{x}^\mu\dot{x}^\nu &= 0 \\ \ddot{x}^\mu + \text{terms proportional to } \dot{x} &= 0 \ . \end{aligned} \tag{2.47}$$

Therefore $\dot{x}^\mu = 0$, $\ddot{y} = 0$ is a solution of the geodesic equation, and it describes motion along the coordinate lines of y.

REMARKS:

(a) In the case of the two-sphere, with its metric $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$, this translates into the familiar statement that the great circles, the coordinate lines of $y = \theta$, are geodesics.

(b) The result is also valid when y is a timelike coordinate. For example, consider a spacetime with coordinates (t, x^i) and metric

$$ds^2 = -dt^2 + a(t)^2 g_{ij}(x)dx^i dx^j \tag{2.48}$$

(this describes a space-time with spatial metric $g_{ij}(x)dx^i dx^j$ and a time-dependent radius $a(t)$; in particular, such a space-time metric can describe an expanding universe in cosmology - see section 15). In such a spacetime, there is, according to the above result, a privileged class of freely falling (i.e. geodesic) observers, namely those that stay at fixed values of the spatial coordinates x^i. For such observers, the coordinate-time t coincides with their proper time τ.

3. Using the Euler-Lagrange equations to determine the Christoffel symbols

Finally, the Euler-Lagrange form of the geodesic equations frequently provides the most direct way of calculating Christoffel symbols - by comparing the Euler-Lagrange equations with the expected form of the geodesic equation in terms of Christoffel symbols. Thus you derive the Euler-Lagrange equations, write them in the form

$$\ddot{x}^\mu + \text{terms proportional to } \dot{x}\dot{x} = 0 \quad , \tag{2.49}$$

and compare with the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu_{\ \nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0 \tag{2.50}$$

to read off the $\Gamma^\mu_{\ \nu\lambda}$.

<u>REMARKS:</u>

(a) Careful - in this and similar calculations beware of factors of 2:

$$\Gamma^\mu_{\ \nu\lambda}\dot{x}^\nu\dot{x}^\lambda = \Gamma^\mu_{\ 11}(\dot{x}^1)^2 + 2\Gamma^\mu_{\ 12}\dot{x}^1\dot{x}^2 + \dots \tag{2.51}$$

(b) For example, once again in the case of the two-sphere, for the θ-equation one has

$$\begin{aligned} \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} &= 2\ddot{\theta} \\ \frac{\partial\mathcal{L}}{\partial\theta} &= 2\sin\theta\cos\theta\dot{\phi}^2 \ . \end{aligned} \tag{2.52}$$

Comparing the variational equation

$$\ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0 \tag{2.53}$$

with the geodesic equation

$$\ddot{\theta} + \Gamma^\theta_{\ \theta\theta}\dot{\theta}^2 + 2\Gamma^\theta_{\ \theta\phi}\dot{\theta}\dot{\phi} + \Gamma^\theta_{\ \phi\phi}\dot{\phi}^2 = 0 \ , \tag{2.54}$$

one can immediately read off that

$$\begin{aligned} \Gamma^\theta_{\ \phi\phi} &= -\sin\theta\cos\theta \\ \Gamma^\theta_{\ \theta\theta} &= \Gamma^\theta_{\ \theta\phi} = 0 \ . \end{aligned} \tag{2.55}$$

2.5 Conserved Charges and (a first encounter with) Killing Vectors

In the previous section we have seen that cyclic coordinates, i.e. coordinates the metric does not depend on, lead to conserved charges, as in (2.40). As nice and useful as this may be (and it is nice and useful), it is obvioulsy somewhat unsatisfactory because it is an explicitly coordinate-dependent statement: the metric may well be independent of one coordinate in some coordinate system, but if one now performs a coordinate transformation which depends on that coordinate, then in the new coordinate system the metric will typically depend on all the new coordinates. Nevertheless,

- the statement that a metric has a certain symmetry (a translational symmetry in the first coordinate system) should be coordinate-independent, and
- thus there should be a corresponding first integral of the geodesic equation in any coordinate system.

To see how this works, let us reconsider the situation discussed in the previous section, namely a metric which in some coordinate system, we will now call it $\{y^\mu\}$, has components $g_{\mu\nu}$ which are independent of y^1, say. Translation invariance of the geodesic Lagrangian is the statement that the Lagrangian is invariant under the infinitesimal variation $\delta y^1 = \epsilon$, $\delta y^\mu = 0$ otherwise, and via Noether's theorem this leads to a conserved charge $g_{1\mu}\dot{y}^\mu$, as in (2.40).

Now we ask ourselves what this statement corresponds to in another coordinate system. Note that in the y-coordinates, invariance is the statement that the metric is invariant under the (infinitesimal) coordinate transformation $y^1 \to y^1 + \epsilon$ or $\delta y^1 = \epsilon$, $\delta y^\mu = 0$ otherwise,

$$\delta g_{\mu\nu} \equiv \partial_{y^1} g_{\mu\nu} = 0 \quad . \tag{2.56}$$

It is then clear that in another coordinate system, infinitesimal y^1-translations must also correspond to some infinitesimal coordinate transformation (but not necessarily just a translation),

$$\delta x^\alpha = \epsilon V^\alpha(x) \quad . \tag{2.57}$$

In particular, if (as in the above example) in y-coordinates V^μ has the components $V^1 = 1$, $V^\mu = 0$ otherwise, then in any other coordinate system one has

$$\delta x^\alpha = (\partial x^\alpha/\partial y^\mu)\delta y^\mu = \epsilon(\partial x^\alpha/\partial y^1) \tag{2.58}$$

so that

$$V^\alpha = J^\alpha_1 \tag{2.59}$$

is just the corresponding column of the Jacobi matrix.

In order to determine how to characterise the translational symmetry (2.56) of the metric in an arbitrary coordinate system, we will now proceed in two (as it turns out ultimately equivalent) ways.

1. We can investigate directly, under which conditions on the V^α the transformation (2.57) leads to an invariance of the Lagrangian (2.2). Using

   $$\delta\dot{x}^\alpha = \dot{x}^\gamma \partial_\gamma V^\alpha \tag{2.60}$$

   one straightforwardly finds

   $$\delta\left(g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\right) = \epsilon\delta_V g_{\alpha\beta} \tag{2.61}$$

   where

   $$\delta_V g_{\alpha\beta} = V^\gamma\partial_\gamma g_{\alpha\beta} + (\partial_\alpha V^\gamma)g_{\gamma\beta} + (\partial_\beta V^\gamma)g_{\alpha\gamma} \tag{2.62}$$

   Thus the condition for the infinitesimal transformation (2.57) to leave the Lagrangian invariant is

   $$\delta_V g_{\alpha\beta} = 0 \quad . \tag{2.63}$$

Noether's theorem then leads to the corresponding conserved charge

$$Q_V = p_\alpha V^\alpha = g_{\alpha\beta} V^\alpha \dot{x}^\beta \quad . \tag{2.64}$$

Note that for constant components V^α, (2.63) is simply the statement that the metric is constant in the direction V, $V^\gamma \partial_\gamma g_{\alpha\beta} = 0$.

2. Alternatively, we can determine the variation $\delta_V g_{\alpha\beta}$ of the components $g_{\alpha\beta}$ of the metric in x-coordinates from the variation (2.56) of the components $g_{\mu\nu}$ of the metric in y-coordinates by demanding that under a coordinate transformation the variation (2.56) of the metric transforms like the metric. Since we know how the metric transforms (1.51), and we also know how ∂_{y^1} transforms,

$$g_{\mu\nu} = J^\alpha_\mu J^\beta_\nu g_{\alpha\beta} \quad , \quad \partial_{y^1} = (\partial_{y^1} x^\alpha)\partial_\alpha \equiv J^\alpha_1 \partial_\alpha \equiv V^\alpha \partial_\alpha \tag{2.65}$$

we find the condition

$$\begin{aligned} \partial_{y^1} g_{\mu\nu} &= J^\gamma_1 \partial_\gamma (J^\alpha_\mu J^\beta_\nu g_{\alpha\beta}) \stackrel{!}{=} J^\alpha_\mu J^\beta_\nu \delta_V g_{\alpha\beta} \\ \Leftrightarrow \quad \delta_V g_{\alpha\beta} &= J^\mu_\alpha J^\nu_\beta J^\gamma_1 \partial_\gamma (J^\delta_\mu J^\epsilon_\nu g_{\delta\epsilon}) \quad . \end{aligned} \tag{2.66}$$

In order to disentangle this, one can make use of identities such as

$$J^\gamma_1 \partial_\gamma J^\delta_\mu = \partial_1 J^\delta_\mu = \partial_\mu J^\delta_1 = \partial_\mu V^\delta = J^\alpha_\mu \partial_\alpha V^\delta \tag{2.67}$$

to show that this expression for $\delta_V g_{\alpha\beta}$ is identical to that given in (2.62).

All of this may seem a bit ham-handed at this point, and indeed it is. However, we will see later how these results can be written and understood in a much more pleasing and covariant way. In particular, we will see in section 4.5 how to write (2.62) in a way that makes it completely manifest that it transforms like the metric under coordinate transformations. Moreover, we will discover in section 6 that (2.62) is a special case of the *Lie derivative* of a tensor field. Continuous symemtries of a metric correspond to vector fields along which the Lie derivative of the metric vanishes. Such vectors are known as *Killing vectors*, and are thus vectors V^α satisfying the *Killing equation* (2.63),

$$L_V g_{\alpha\beta} \equiv \delta_V g_{\alpha\beta} = 0 \quad . \tag{2.68}$$

2.6 The Newtonian Limit

We saw that the 10 components of the metric $g_{\mu\nu}$ appear to play the role of potentials for the gravitational force. In order to substantiate this, and to show that in an appropriate limit this setting is able to reproduce the Newtonian results, we now want to find the relation of these potentials to the Newtonian potential, and the relation between the geodesic equation and the Newtonian equation of motion for a particle moving in a gravitational field.

First let us determine the conditions under which we might expect the general relativistic equation of motion (namely the non-linear coupled set of partial differential geodesic equations) to reduce to the linear equation of motion

$$\frac{d^2}{dt^2}\vec{x} = -\vec{\nabla}\phi \tag{2.69}$$

of Newtonian mechanics, with ϕ the gravitational potential, e.g.

$$\phi = -\frac{GM}{r} \quad . \tag{2.70}$$

Thus we are trying to characterise the circumstances in which we know and can trust the validity of Newton's equations, such as those provided e.g. by the gravitational field of the earth or the sun, the gravitational fields in which Newton's laws were discovered and tested. Two of these are fairly obvious:

1. Weak Fields: our first plausible assumption is that the gravitational field is in a suitable sense sufficiently weak. We will need to make more precise by what we mean by this, and we will come back to this below.

2. Slow Motion: our second, equally reasonable and plausible, assumption is that the test particle moves at speeds at which we can neglect special relativistic effects, so "slow" should be taken to mean that its velocity is small compared to the velocity of light.

Interestingly, it turns out that one more condition is required. Note that the gravitational fields we have access to are not only quite weak but also only very slowly varying in time, and we will add this condition,

3. Stationary Fields: we will assume that the gravitational field does not vary significantly in time (over the time scale probed by our test particle).

The very fact that we have to add this condition in order to find Newton's equations (as will be borne out by the calculations below) is interesting in its own right, because it also shows that general relativity predicts phenomena deviating from the Newtonian picture even for weak fields, provided that they vary sufficiently rapidly (e.g. quickly oscillating fields), and one such phenomenon is that of gravitational waves (see section 19).

Now, having formulated in words the conditions that we wish to impose, we need to translate these conditions into equations that we can then use in conjunction with the geodesic equation.

1. In order to define a notion of weak fields, we need to keep in mind that this is not a coordinate-independent statement since we can simulate arbitrarily strong

gravitational fields even in Minkowski space by going to suitably accelerated coordinates, and therefore a "weak field" condition will be a condition not only on the metric but also on the choice of coordinates. Thus we assume that we can choose coordinates $\{x^\mu\} = \{t, x^i\}$ in such a way that *in these coordinates* the metric differs from the standard constant Minkowski metric $\eta_{\mu\nu}$ only by a small amount,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{2.71}$$

where we will implement "by a small amount" in the calculations below by dropping all terms that are at least quadratic in $h_{\mu\nu}$.

2. The second condition is obviously (with the coordinates chosen above) $dx^i/dt \ll 1$ or, expressed in terms of proper time,

$$\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau} \quad . \tag{2.72}$$

3. The third condition of stationarity we implement simply by considering time-independent fields,

$$g_{\alpha\beta,0} = 0 \quad \Rightarrow \quad h_{\alpha\beta,0} = 0 \ . \tag{2.73}$$

Now we look at the geodesic equation. The condition of slow motion (condition 2) implies that the geodesic equation can be approximated by

$$\ddot{x}^\mu + \Gamma^\mu_{\ 00}\dot{t}^2 = 0 \ . \tag{2.74}$$

Stationarity (condition 3) tells us that

$$\Gamma^\mu_{\ 00} = -\tfrac{1}{2}g^{\mu\nu}\partial_\nu g_{00} = -\tfrac{1}{2}g^{\mu i}\partial_i g_{00} \ . \tag{2.75}$$

From the weak field condition (condition 1), which allows us to write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ \Rightarrow \ g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \ , \tag{2.76}$$

where

$$h^{\mu\nu} = \eta^{\mu\lambda}\eta^{\nu\rho}h_{\lambda\rho} \ , \tag{2.77}$$

we learn that

$$\Gamma^\mu_{\ 00} = -\tfrac{1}{2}\eta^{\mu i}\partial_i h_{00} \ , \tag{2.78}$$

so that the relevant Christoffel symbols are

$$\Gamma^0_{\ 00} = 0 \ , \qquad \Gamma^i_{\ 00} = -\tfrac{1}{2}\partial^i h_{00} \ . \tag{2.79}$$

Thus the geodesic equation splits into

$$\begin{aligned} \ddot{t} &= 0 \\ \ddot{x}^i &= \tfrac{1}{2}\partial_i h_{00}\dot{t}^2 \ . \end{aligned} \tag{2.80}$$

As the first of these just says that $\dot{t}$ is constant, we can use this in the second equation to convert the τ-derivatives into derivatives with respect to the coordinate time t. Hence we obtain

$$\frac{d^2x^i}{dt^2} = \tfrac{1}{2}h_{00,i} \tag{2.81}$$

(the spatial index i in this expression is raised or lowered with the Kronecker symbol, $\eta^{ik} = \delta^{ik}$). Comparing this with the Newtonian equation (2.69),

$$\frac{d^2x^i}{dt^2} = -\phi_{,i} \tag{2.82}$$

leads us to the identification $h_{00} = -2\phi$, with the constant of integration absorbed into an arbitrary constant term in the gravitational potential. By relating this back to $g_{\alpha\beta}$,

$$g_{00} = -(1 + 2\phi) \ . \tag{2.83}$$

we find the sought-for relation between the Newtonian potential and the space-time metric. For the gravitational field of isolated systems, it makes sense to choose the integration constant in such a way that the potential goes to zero at infinity, and this choice also ensures that the metric approaches the flat Minkowski metric at infinity.

Restoring the appropriate units, in particular a factor of c^2, one finds that $\phi/c^2 \sim 10^{-9}$ on the surface of the earth, 10^{-6} on the surface of the sun (see section 11.5 for some more details), so that the distortion in the space-time geometry produced by gravitation is in general quite small (justifying our approximations).

Note that it does not make sense in this approximation to inquire about the other components of the metric. As we have seen, a slowly moving particle in weak stationary gravitational field is not sensitive to them, and hence can also not be used to probe or determine these components. Later on, we will determine the exact solution for the metric outside a spherically symmetric mass distribution (the Schwarzschild metric), and from this one can then also read off that the leading correction to the flat metric arises from the 00-component of the metric, but one can then also determine the subleading "post-Newtonian" corrections to the gravitational field.

2.7 The Gravitational Red-Shift

The gravitational red-shift (i.e. the fact that photons lose or gain energy when rising or falling in a gravitational field) is a consequence of the Einstein Equivalence Principle (and therefore also provides an experimental test of the Einstein Equivalence Principle).

It is clear from the expression $d\tau^2 = -g_{\mu\nu}(x)dx^\mu dx^\nu$ that e.g. the rate of clocks is affected by the gravitational field. However, as everything is affected in the same way by gravity it is impossible to measure this effect locally. In order to find an observable effect, one needs to compare data from two different points in a gravitational potential.

The situation we could consider is that of two observers A and B moving on worldlines (paths) γ_A and γ_B, A sending light signals to B. In general the frequency, measured in the observers rest-frame at A (or in a locally inertial coordinate system there) will differ from the frequency measured by B upon receiving the signal.

In order to separate out Doppler-like effects due to relative velocities, we consider two observers A and B at rest radially to each other, at radii r_A and r_B, in a stationary spherically symmetric gravitational field. This means that the metric depends only on a radial coordinate r and we can choose it to be of the form

$$ds^2 = g_{00}(r)dt^2 + g_{rr}(r)dr^2 + r^2 d\Omega^2 \quad , \tag{2.84}$$

where $d\Omega^2$ is the standard volume element on the two-sphere (see section 11 for a more detailed justification of this ansatz for the metric).

Observer A sends out light of a given frequency ν_A, say n pulses per proper time unit $\Delta\tau_A$. Observer B receives these n pulses in his proper time $\Delta\tau_B$ and interprets this as a frequency ν_B. Thus the relation between the frequency ν_A emitted at A and the frequency ν_B observed at B is

$$\frac{\nu_A}{\nu_B} = \frac{\Delta\tau_B}{\Delta\tau_A} \quad . \tag{2.85}$$

I will now give two arguments to show that this ratio depends on the metric (i.e. the gravitational field) at r_A and r_B through

$$\frac{\nu_A}{\nu_B} = (g_{00}(r_B)/g_{00}(r_A))^{1/2} \quad . \tag{2.86}$$

1. The first argument is essentially one based on geometric optics (and is best accompanied by drawing a (1+1)-dimensional space-time diagram of the light rays and worldlines of the observers).

   The geometry of the situation dictates that the coordinate time intervals recorded at A and B are equal, $\Delta t_A = \Delta t_B$ as nothing in the metric actually depends on t. In equations, this can be seen as follows. First of all, the equation for a radial light ray is

   $$-g_{00}(r)dt^2 = g_{rr}(r)dr^2 \quad , \tag{2.87}$$

   or

   $$\frac{dt}{dr} = \pm\left(\frac{g_{rr}(r)}{-g_{00}(r)}\right)^{1/2} \quad . \tag{2.88}$$

   From this we can calculate the coordinate time for the light ray to go from A to B. Say that the first light pulse is emitted at point A at time $t(A)_1$ and received at B at coordinate time $t(B)_1$. Then

   $$t(B)_1 - t(A)_1 = \int_{r_A}^{r_B} dr(-g_{rr}(r)/g_{00}(r))^{1/2} \tag{2.89}$$

But the right hand side obviously does not depend on t, so we also have

$$t(B)_2 - t(A)_2 = \int_{r_A}^{r_B} dr(-g_{rr}(r)/g_{00}(r))^{1/2} \tag{2.90}$$

where t_2 denotes the coordinate time for the arrival of the n-th pulse. Therefore,

$$t(B)_1 - t(A)_1 = t(B)_2 - t(A)_2 \ , \tag{2.91}$$

or

$$t(A)_2 - t(A)_1 = t(B)_2 - t(B)_1 \ , \tag{2.92}$$

as claimed. Thus the coordinate time intervals recorded at A and B between the first and last pulse are equal. However, to convert this to proper time, we have to multiply the coordinate time intervals by an r-dependent function,

$$\Delta\tau_{A,B} = (-g_{\mu\nu}(r_{A,B})\frac{dx^\mu}{dt}\frac{dx^\nu}{dt})^{1/2}\Delta t_{A,B} \ , \tag{2.93}$$

and therefore the proper time intervals will not be equal. For observers at rest, $dx^i/dt = 0$, one has

$$\Delta\tau_{A,B} = (-g_{00}(r_{A,B}))^{1/2}\Delta t_{A,B} \ . \tag{2.94}$$

Since $\Delta t_A = \Delta t_B$, (2.86) now follows from (2.85).

2. The second argument uses the null geodesic equation, in particular the conserved quantity associated to time-translations (recall that we have assumed that the metric (2.84) is time-independent), as well as a somewhat more covariant looking, but equivalent, notion of frequency.

   First of all, let the light ray be described the wave vector k^μ. In special relativity, we would parametrise this as $k^\mu = (\omega, \vec{k})$ with $\omega = 2\pi\nu$ the frequency. This is the frequency observed by an inertial observer at rest, with 4-velocity $u^\mu = (1,0,0,0)$. A Lorentz-invariant, and thus in our context now coordinate-independent, notion of the frequency as measured by an observer with velocity u^μ is thus

$$\omega = -u^\mu k_\mu \ . \tag{2.95}$$

   This includes as special cases the relativistic Doppler effect (where one compares ω with $\bar{\omega} = -\bar{u}^\mu k_\mu$, $\bar{u}^\mu$ the tangent to the world line of a boosted observer), as well as the gravitational red-shift we want to discuss here.

   A static observer in the spherically-symmetric and static gravitational field (2.84) is described by the 4-velocity

$$u^\mu = (u^0, 0, 0, 0) \qquad g_{\mu\nu}u^\mu u^\nu = g_{00}(u^0)^2 = -1 \ . \tag{2.96}$$

   Thus for the static observer at $r = r_A$, say, one has

$$u_A^0 = (-g_{00}(r_A))^{-1/2} \tag{2.97}$$

(and likewise for the observer at $r = r_B$). The wave vector k^μ is a null tangent vector, $k^\mu k_\mu = 0$, to a null geodesic corresponding to the Lagrangian

$$\mathcal{L} = \tfrac{1}{2} g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = \tfrac{1}{2} g_{00}(r)\dot{t}^2 + \ldots \tag{2.98}$$

Since the metric is time-independent, there is (cf. the discussion in section 2.4) the corresponding conserved quantity

$$E = -\frac{\partial \mathcal{L}}{\partial \dot{t}} = -g_{00}(r)\dot{t} \tag{2.99}$$

(the minus sign serving only to make this quantity positive for $\dot{t} > 0$). Then one finds that the frequency measured by the stationary observer at $r = r_A$ is

$$\begin{aligned} \omega_A = -u_A^\mu k_\mu &= -\left(-g_{00}(r_A)\right)^{-1/2} k_0 = -\left(-g_{00}(r_A)\right)^{-1/2} g_{0\mu}(r_A)\dot{x}^\mu \\ &= -\left(-g_{00}(r_A)\right)^{-1/2} g_{00}(r_A)\dot{t} = E\left(-g_{00}(r_A)\right)^{-1/2} \end{aligned} \tag{2.100}$$

Since E is a conserved quantity, i.e. the same for the light ray at $r = r_A$ or $r = r_B$, one sees that $\omega_A/\omega_B = \nu_A/\nu_B$ is given by (2.86), as claimed.

Note also that this derivation shows that the relation between ω and E is exactly like the relation (2.94) between $(\Delta\tau)^{-1}$ and $(\Delta t)^{-1}$, which provides us with an interpreteation of the conserved quantity E for a massless particle / photon: it is the frequency measured with respect to coordinate time (as the momentum conjugate to the time-coordinate t this should not be too surprising).

Using the Newtonian approximation, (2.86) becomes

$$\frac{\nu_A}{\nu_B} \sim 1 + \phi(r_B) - \phi(r_A) \ , \tag{2.101}$$

or

$$\frac{\nu_A - \nu_B}{\nu_B} = \frac{GM(r_B - r_A)}{r_A r_B} \tag{2.102}$$

Note that, for example, for $r_B > r_A$ one has $\nu_B < \nu_A$ so that, as expected, a photon loses energy when rising in a gravitational field.

This result can also be deduced from energy conservation. A local inertial observer at the emitter A will see a change in the internal mass of the emitter $\Delta m_A = -h\nu_A$ when a photon of frequency of ν_A is emitted. Likewise, the absorber at point B will experience an increase in inertial mass by $\Delta m_B = h\nu_B$. But the total internal plus gravitational potential energy must be conserved. Thus

$$0 = \Delta m_A(1 + \phi(r_A)) + \Delta m_B(1 + \phi(r_B)) \ , \tag{2.103}$$

leading to

$$\frac{\nu_A}{\nu_B} = \frac{1 + \phi(r_B)}{1 + \phi(r_A)} \sim 1 + \phi(r_B) - \phi(r_A) \ , \tag{2.104}$$

as before. This "derivation" (in quotes, because we are wildly mixing Newtonian gravity, special relativity and quantum mechanics - do take this "derivation" with an appropriately sized grain of salt, please) shows that gravitational red-shift experiments test the Einstein Equivalence Principle in its strong form, in which the term 'laws of nature' is not restricted to mechanics (inertial = gravitational mass), but also includes quantum mechanics in the sense that it tests if in an inertial frame the relation between photon energy and frequency is unaffected by the presence of a gravitational field.

While difficult to observe directly (by looking at light form the sun), this prediction has been verified in the laboratory, first by Pound and Rebka (1960), and subsequently, with one percent accuracy, by Pound and Snider in 1964 (using the Mössbauer effect).

Let us make some rough estimates of the expected effect. We first consider light reaching us (B) from the sun (A). In this case, we have $r_B \gg r_A$, where r_A is the radius of the sun, and (also inserting a so far suppressed factor of c^2) we obtain

$$\frac{\nu_A - \nu_B}{\nu_B} = \frac{GM(r_B - r_A)}{c^2 r_A r_B} \simeq \frac{GM}{c^2 r_A} \quad . \tag{2.105}$$

Using the approximate values

$$\begin{aligned} r_A &\simeq 0.7 \times 10^6 \text{ km} \\ M_{\text{sun}} &\simeq 2 \times 10^{33} \text{ g} \\ G &\simeq 7 \times 10^{-8} \text{ g}^{-1}\text{cm}^3\text{s}^{-2} \\ Gc^{-2} &\simeq 7 \times 10^{-29} \text{ g}^{-1}\text{cm} = 7 \times 10^{-34} \text{ g}^{-1}\text{km} \quad , \end{aligned} \tag{2.106}$$

one finds

$$\frac{\Delta\nu}{\nu} \simeq 2 \times 10^{-6} \quad . \tag{2.107}$$

In principle, such a frequency shift should be observable. In practice, however, the spectral lines of light emitted by the sun are strongly effected e.g. by convection in the atmosphere of the sun (Doppler effect), and this makes it difficult to measure this effect with the required precision.

In the Pound-Snider experiment, the actual value of $\Delta\nu/\nu$ is much smaller. In the original set-up one has $r_B - r_A \simeq 20\text{m}$ (the distance from floor to ceiling of the laboratory), and $r_A = r_{\text{earth}} \simeq 6.4 \times 10^6\text{m}$, leading to

$$\frac{\Delta\nu}{\nu} \simeq 2.5 \times 10^{-15} \quad . \tag{2.108}$$

However, here the experiment is much better controlled, and the gravitational red-shift was verified with 1% accuracy.

2.8 Locally Inertial and Riemann Normal Coordinates

Central to our initial discussion of gravity was the Einstein Equivalence Principle which postulates the existence of locally inertial (or freely falling) coordinate systems in which locally at (or around) a point the effects of gravity are absent. Now that we have decided that the arena of gravity is a general metric space-time, we should establish that such coordinate systems indeed exist. Looking at the geodesic equation, it it is clear that 'absence of gravitational effects' is tantamount to the existence of a coordinate system $\{\xi^A\}$ in which at a given point p the metric is the Minkowski metric, $g_{AB}(p) = \eta_{AB}$ and the Christoffel symbol is zero, $\Gamma^A_{BC}(p) = 0$. Owing to the identity

$$g_{\mu\nu},_\lambda = \Gamma_{\mu\nu\lambda} + \Gamma_{\nu\mu\lambda} \quad , \tag{2.109}$$

the latter condition is equivalent to $g_{AB},_C(p) = 0$. I will sketch three arguments establishing the existence of such coordinate systems, each one having its own virtues and providing its own insights into the issue.

Actually it is physically plausible (and fortuitously moreover true) that one can always find coordinates which embody the equivalence principle in the stronger sense that the metric is the flat metric η_{AB} and the Christoffel symbols are zero not just at a point but along the entire worldline of an inertial (freely falling) observer, i.e. along a geodesic. Such coordinates, based on a geodesic rather than on a point, are known as Fermi normal coordinates. The construction is similar to that of Riemann normal coordinates (based at a point) to be discussed below.

1. Direct Construction

   We know that given a coordinate system $\{\xi^A\}$ that is inertial at a point p, the metric and Christoffel symbols at p in a new coordinate system $\{x^\mu\}$ are determined by (1.33,1.40). Conversely, we will now see that knowledge of the metric and Christoffel symbols at a point p is sufficient to construct a locally inertial coordinate system at p.

   Equation (1.40) provides a second order differential equation in some coordinate system $\{x^\mu\}$ for the inertial coordinate system $\{\xi^A\}$, namely

$$\frac{\partial^2 \xi^A}{\partial x^\nu \partial x^\lambda} = \Gamma^\mu_{\ \nu\lambda} \frac{\partial \xi^A}{\partial x^\mu} \quad . \tag{2.110}$$

   By a general theorem, a local solution around p with given initial conditions $\xi^A(p)$ and $(\partial\xi^A/\partial x^\mu)(p)$ is guaranteed to exist. In terms of a Taylor series expansion around p one has

$$\xi^A(x) = \xi^A(p) + \frac{\partial \xi^A}{\partial x^\mu}|_p (x^\mu - p^\mu) + \frac{1}{2}\frac{\partial \xi^A}{\partial x^\mu}|_p \Gamma^\mu_{\ \nu\lambda}(p)(x^\nu - p^\nu)(x^\lambda - p^\lambda) + \ldots \tag{2.111}$$

   It follows from (1.33) that the metric at p in the new coordinate system is

$$g_{AB}(p) = g_{\mu\nu}(p) \frac{\partial x^\mu}{\partial \xi^A}|_p \frac{\partial x^\nu}{\partial \xi^B}|_p \quad . \tag{2.112}$$

Since a symmetric matrix (here the metric at the point p) can always be diagonalised by a similarity transformation, for an appropriate choice of initial condition $(\partial\xi^A/\partial x^\mu)(p)$ one can arrange that $g_{AB}(p)$ is the standard Minkowski metric, $g_{AB}(p) = \eta_{AB}$.

With a little bit more work it can also be shown that in these coordinates (2.111) one also has $g_{AB},_C(p) = 0$. Thus this is indeed an inertial coordinate system at p.

As the matrix $(\partial\xi^A/\partial x^\mu)(p)$ which transforms the metric at p into the standard Minkowski form is only unique up to Lorentz transformations, overall (counting also the initial condition $\xi^A(p)$) a locally inertial coordinate system is unique only up to Poincaré transformations - an unsurprising result.

2. Geodesic (or Riemann Normal) Coordinates

   A slightly more insightful way of constructing a locally inertial coordinate system, rather than by directly solving the relevant differential equation, makes use of geodesics at p. Recall that in Minkowski space the metric takes the simplest possible form in coordinates whose coordinate lines are geodesics. One might thus suspect that in a general metric space-time the metric will also (locally) look particularly simple when expressed in terms of such *geodesic coordinates*. Since locally around p we can solve the geodesic equation with four linearly independent initial conditions, we can assume the existence of a coordinate system $\{\xi^A\}$ in which the coordinate lines are geodesics $\xi^A(\tau) = \xi^A\tau$. But this means that $\ddot{\xi}^A = 0$. Hence the geodesic equation reduces to

$$\Gamma^A_{BC}\dot{\xi}^B\dot{\xi}^C = 0 \quad . \tag{2.113}$$

   As at p the $\dot{\xi}^A$ were chosen to be linearly independent, this implies $\Gamma^A_{BC}(p) = 0$, as desired. It is easy to see that the coordinates ξ^A can also be chosen in such a way that $g_{AB}(p) = \eta_{AB}$ (by choosing the four directions at p to be orthonormal unit vectors).

   As an example, consider the standard metric $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$ on the two-sphere. Any point is as good as any other point, and one can construct an inertial coordinate system at the north pole $\theta = 0$ in terms of geodesics shot off from the north pole into the $\phi = 0$ (ξ^1) and $\phi = \pi/2$ (ξ^2) directions. The affine parameter along a great circle (geodesic) connecting the north pole to a point (θ, ϕ) is θ, and thus θ is also the geodesic distance, and the coordinates of the point (θ, ϕ) are $(\xi^1 = \theta\cos\phi, \xi^2 = \theta\sin\phi)$. In particular, the north pole is the origin $\xi^1 = \xi^2 = 0$. Calculating the metric in these new components one finds (exercise! see also e.g. Hartle's *Gravity*) that $g_{AB}(\xi) = \delta_{AB} + \mathcal{O}(\xi^2)$, so that $g_{AB}(\xi = 0) = \delta_{AB}$ and $g_{AB,C}(\xi = 0) = 0$, as required. Note also that one could have guessed these coordinates from the fact that near $\theta = 0$ the metric is $d\theta^2 + \theta^2 d\phi^2$, which is the Euclidean metric in polar coordinates $(\theta\cos\phi, \theta\sin\phi)$.

3. A Numerological Argument

This is my favourite argument because it requires no calculations and at the same time provides additional insight into the nature of curved space-times.

Assuming that the local existence of solutions to differential equations is guaranteed by some mathematical theorems, it is frequently sufficient to check that one has enough degrees of freedom to satisfy the desired initial conditions (one may also need to check integrability conditions). In the present context, this argument is useful because it also reveals some information about the 'true' curvature hidden in the second derivatives of the metric. It works as follows:

Consider a Taylor expansion of the metric around p in the sought-for new coordinates. Then the metric at p will transform with the matrix $(\partial x^\mu/\partial\xi^A)(p)$. This matrix has $(4\times 4) = 16$ independent components, precisely enough to impose the 10 conditions $g_{AB}(p) = \eta_{AB}$ up to Lorentz transformations.

The derivative of the metric at p, $g_{AB,C}(p)$, will appear in conjunction with the second derivative $\partial^2 x^\mu/\partial\xi^A\partial\xi^B$. The $4\times(4\times 5)/2 = 40$ coefficients are precisely sufficient to impose the 40 conditions $g_{AB,C}(p) = 0$.

Now let us look at the second derivatives of the metric. $g_{AB,CD}$ has $(10\times 10) = 100$ independent components, while the third derivative of $x^\mu(\xi)$ at p, $\partial^3 x^\mu/\partial\xi^A\partial\xi^B\partial\xi^C$ has $4\times(4\times 5\times 6)/(2\times 3) = 80$ components. Thus 20 linear combinations of the second derivatives of the metric at p cannot in general be set to zero by a coordinate transformation. Thus these encode the information about the real curvature at p. This agrees nicely with the fact that the Riemann curvature tensor we will construct later turns out to have precisely 20 independent components.

Repeating this argument in space-time dimension n, one finds that the number of 2nd derivatives of the metric modulo coordinate transformations is

$$\left(\frac{n(n+1)}{2}\right)^2 - n\frac{n(n+1)(n+2)}{6} = \frac{1}{12}n^2(n^2-1) \ . \qquad (2.114)$$

Again this turns out to agree with the number of independent components of the curvature tensor in n dimensions.

Note: At this point in the course I find it useful to develop in parallel (and suggest to read in parallel)

- the more formal material on tensor analysis in sections 3 - 8, say,
- and a detailed discussion of the basic properties of the Schwarzschild metric (sections 11.5 - 13),

since much of the latter (in particular geodesics, solar system tests of general relativity, even the issues that arise in connection with the Schwarzschild radius) can be understood just on the basis of what has been done so far (if, for the time being, one accepts on faith that the Schwarzschild metric is the unique spherically symmetric vacuum solution of the Einstein field equations). Not only is this an interesting and physically relevant application of the machinery developed so far, it also provides an appropriate balance between physics and formalism in the lectures.

3 Tensor Algebra

3.1 From the Einstein Equivalence Principle to the Principle of General Covariance

The Einstein Equivalence Principle tells us that the laws of nature (including the effects of gravity) should be such that in an inertial frame they reduce to the laws of Special Relativity (SR). As we have seen, this can be implemented by transforming the laws of SR to arbitrary coordinate systems and declaring that these be valid for arbitrary coordinates and metrics.

However, this is a tedious method in general (e.g. to obtain the correct form of the Maxwell equations in the presence of gravity) and not particularly enlightning. We will thus replace the Einstein Equivalence Principle by the closely related *Principle of General Covariance* PGC:

> A physical equation holds in an arbitrary gravitational field if
>
> 1. the equation holds in the absence of gravity, i.e. when $g_{\mu\nu} = \eta_{\mu\nu}$, $\Gamma^{\mu}_{\ \nu\lambda} = 0$, and
> 2. the equation is *generally covariant*, i.e. preserves its form under a general coordinate transformation.

Concretely, the 2nd condition means the following: assume that you have some physical equation that in some coordinate system takes the form $T = 0$, where $T = 0$ could be some multi-component (thus T is adorned with various indices) differential equation. Now perform a coordinate transformation $x \to y(x)$, and assume that the new object T' has the form

$$T' = (\ldots)T + \text{junk} \tag{3.1}$$

where the term in brackets is some invertible matrix or operator. Then clearly the presence of the junk-terms means that the equation $T' = 0$ is not equivalent to the equation $T = 0$. An example of an object that transform in this way is, as we have seen, the Christoffel symbols. On the other hand, if these junk terms are absent, so that we have

$$T' = (\ldots)T \tag{3.2}$$

then clearly $T' = 0$ if and only if $T = 0$, i.e. the equation is satisfied in one coordinate system if and only if it is satisfied in any other (or all) coordinate systems. This is the kind of equation that embodies general covariance, and again we have already seen an example of such an equation, namely the geodesic equation, where the term $(\ldots)$ in brackets is just the Jacobi matrix. Thus, to be more concrete, we can replace the 2nd condition above by

2' the equation is *generally covariant*, i.e. it is satisfied in one coordinate system iff it is satisfied in all coordinate systems.

Let us now establish the above statement, namely that the Einstein equivalence principle implies that an equation that satisfies the conditions 1 and 2 (or 2') is valid in an arbitrary graviational field:

- consider some equation that satsifies these conditions, and assume that we are in an arbitrary gravitational field;
- condition 2' implies that this equation is true (or satisfied) in all coordinate systems if it is satisfied just in one coordinate system;
- now we know that we can always (locally) construct a freely falling coordinate system in which the effects of gravity are absent;
- the Einstein Equivalence Principle now posits that in such a reference system the physics is that of Minkowski space-time;
- condition 1 means that the equation is true (satisfied) there;
- thus it is valid in all coordinate systems;
- since we started off by considering an arbitrary graviational field, it follows that the equation is now valid in an arbitrary gravitational field, as claimed in the Principle of General Covariance.

Note that general covariance alone is an empty statement since any equation (whether correct or not) can be made generally covariant simply by writing it in an arbitrary coordinate system. It develops its power only when used in conjunction with the Einstein Equivalence Principle as a statement about physics in a gravitational field, namely that by virtue of its general covariance an equation will be true in a gravitational field if it is true in the absence of gravitation.

3.2 Tensors

In order to construct generally covariant equations, we need objects that transform in a simple way under coordinate transformations. The prime examples of such objects are *tensors*.

Scalars

The simplest example of a tensor is a function (or scalar) f which under a coordinate transformation $x^\mu \to y^{\mu'}(x^\mu)$ simply transforms as

$$f'(y(x)) = f(x) \ , \tag{3.3}$$

or $f'(y) = f(x(y))$. One frequently suppresses the argument, and thus writes simply, $f' = f$, expressing the fact that, up to the obvious change of argument, functions are invariant under coordinate transformations.

Vectors

The next simplest case are vectors $V^\mu(x)$ transforming as

$$V'^{\mu'}(y(x)) = \frac{\partial y^{\mu'}}{\partial x^\mu} V^\mu(x) \quad . \tag{3.4}$$

A prime example is the tangent vector $\dot{x}^\mu$ to a curve, for which this transformation behaviour

$$\dot{x}^\mu \rightarrow \dot{y}^{\mu'} = \frac{\partial y^{\mu'}}{\partial x^\mu} \dot{x}^\mu \tag{3.5}$$

is just the familiar one.

It is extremely useful to think of vectors as first order differential operators, via the correspondence

$$V^\mu \Leftrightarrow V := V^\mu \partial_\mu \quad . \tag{3.6}$$

One of the advantages of this point of view is that V is completely invariant under coordinate transformations as the *components* V^μ of V transform inversely to the *basis vectors* ∂_μ. For more on this see the (optional) section on the coordinate-independent interpretation of tensors below.

Covectors

A covector is an object $U_\mu(x)$ which under a coordinate transformation transforms inversely to a vector, i.e. as

$$U'_{\mu'}(y(x)) = \frac{\partial x^\mu}{\partial y^{\mu'}} U_\mu(x) \quad . \tag{3.7}$$

A familiar example of a covector is the derivative $U_\mu = \partial_\mu f$ of a function which of course transforms as

$$\partial_{\mu'} f'(y(x)) = \frac{\partial x^\mu}{\partial y^{\mu'}} \partial_\mu f(x) \quad . \tag{3.8}$$

Covariant 2-Tensors

Clearly, given the above objects, we can construct more general objects which transform in a nice way under coordinate transformations by taking products of them. Tensors in general are objects which transform like (but need not be equal to) products of vectors and covectors.

In particular, a covariant 2-tensor, or (0,2)-tensor, is an object $A_{\mu\nu}$ that transforms under coordinate transformations like the product of two covectors, i.e.

$$A'_{\mu'\nu'}(y(x)) = \frac{\partial x^\mu}{\partial y^{\mu'}} \frac{\partial x^\nu}{\partial y^{\nu'}} A_{\mu\nu}(x) \quad . \tag{3.9}$$

I will from now on use a shorthand notation in which I drop the prime on the transformed object and also omit the argument. In this notation, the above equation would then become

$$A_{\mu'\nu'} = \frac{\partial x^\mu}{\partial y^{\mu'}}\frac{\partial x^\nu}{\partial y^{\nu'}}A_{\mu\nu} \quad . \tag{3.10}$$

We already know one example of such a tensor, namely the metric tensor $g_{\mu\nu}$ (which happens to be a symmetric tensor).

Contravariant 2-Tensors

Likewise we define a contravariant 2-tensor (or a (2,0)-tensor) to be an object $B^{\mu\nu}$ that transforms like the product of two vectors,

$$B^{\mu'\nu'} = \frac{\partial y^{\mu'}}{\partial x^\mu}\frac{\partial y^{\nu'}}{\partial x^\nu}B^{\mu\nu} \quad . \tag{3.11}$$

An example is the inverse metric tensor $g^{\mu\nu}$.

(p,q)-Tensors

It should now be clear how to define a general (p,q)-tensor - as an object $T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}$ with p contravariant and q covariant indices which under a coordinate transformation transforms like a product of p vectors and q covectors,

$$T^{\mu'_1\ldots\mu'_p}_{\nu'_1\ldots\nu'_q} = \frac{\partial y^{\mu'_1}}{\partial x^{\mu_1}}\cdots\frac{\partial y^{\mu'_p}}{\partial x^{\mu_p}}\frac{\partial x^{\nu_1}}{\partial y^{\nu'_1}}\cdots\frac{\partial x^{\nu_q}}{\partial y^{\nu'_q}}T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} \quad . \tag{3.12}$$

Note that, in particular, a tensor is zero (at a point) in one coordinate system if and only if the tensor is zero (at the same point) in another coordinate system.

Thus, any law of nature (field equation, equation of motion) expressed in terms of tensors, say in the form $T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} = 0$, preserves its form under coordinate trasformations and is therefore automatically generally covariant,

$$T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} = 0 \Leftrightarrow T^{\mu'_1\ldots\mu'_p}_{\nu'_1\ldots\nu'_q} = 0 \tag{3.13}$$

An important special example of a tensor is the Kronecker tensor δ^μ_ν. Together with scalars and products of scalars and Kronecker tensors it is the only tensor whose components are the same in all coordinate systems. I.e. if one demands that δ^μ_ν transforms as a tensor, then one finds that it takes the same numerical values in all coordinate systems, i.e. $\delta'^{\mu'}_{\nu'} = \delta^{\mu'}_{\nu'}$. Conversely, if one posits that $\delta'^{\mu'}_{\nu'} = \delta^{\mu'}_{\nu'}$, one can deduce that δ^μ_ν transforms as (i.e. is) a $(1,1)$-tensor.

One comment on terminology: it is sometimes useful to distinguish vectors from vector fields and, likewise, tensors from tensor fields. A vector is then just a vector $V^\mu(x)$ at some point x of space-time whereas a vector field is something that assigns a vector to each point of space-time and, likewise, for tensors and tensor fields.

Important examples of non-tensors are the Christoffel symbols. Another important example is the the ordinary partial derivative of a (p,q)-tensor, $\partial_\lambda T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}$ which is *not* a $(p,q+1)$-tensor unless $p=q=0$. This failure of the partial derivative to map tensors to tensors will motivate us below to introduce a *covariant derivative* which generalises the usual notion of a partial derivative and has the added virtue of mapping tensors to tensors.

3.3 Tensor Algebra

Tensors can be added, multiplied and contracted in certain obvious ways. The basic algebraic operations are the following:

1. Linear Combinations

   Given two (p,q)-tensors $A^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}$ and $B^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}$, their sum

   $$C^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} = A^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} + B^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} \tag{3.14}$$

   is also a (p,q)-tensor.

2. Direct Products

   Given a (p,q)-tensor $A^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}$ and a (p',q')-tensor $B^{\lambda_1\ldots\lambda_{p'}}_{\rho_1\ldots\rho_{q'}}$, their direct product

   $$A^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} B^{\lambda_1\ldots\lambda_{p'}}_{\rho_1\ldots\rho_{q'}} \tag{3.15}$$

   is a $(p+p',q+q')$-tensor,

3. Contractions

   Given a (p,q)-tensor with p and q non-zero, one can associate to it a $(p-1,q-1)$-tensor via contraction of one covariant and one contravariant index,

   $$A^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} \rightarrow B^{\mu_1\ldots\mu_{p-1}}_{\nu_1\ldots\nu_{q-1}} = A^{\mu_1\ldots\mu_{p-1}\lambda}_{\nu_1\ldots\nu_{q-1}\lambda} \quad . \tag{3.16}$$

   This is indeed a $(p-1,q-1)$-tensor, i.e. transforms like one. Consider, for example, a (1,2)-tensor $A^\mu_{\ \nu\lambda}$ and its contraction $B_\nu = A^\mu_{\ \nu\mu}$. Under a coordinate transformation B_ν transforms as a covector:

   $$\begin{aligned} B_{\nu'} &= A^{\mu'}_{\ \nu'\mu'} \\ &= \frac{\partial y^{\mu'}}{\partial x^\mu}\frac{\partial x^\nu}{\partial y^{\nu'}}\frac{\partial x^\lambda}{\partial y^{\mu'}} A^\mu_{\ \nu\lambda} \\ &= \frac{\partial x^\nu}{\partial y^{\nu'}}\delta^\lambda_{\ \mu} A^\mu_{\ \nu\lambda} \\ &= \frac{\partial x^\nu}{\partial y^{\nu'}} A^\mu_{\ \nu\mu} = \frac{\partial x^\nu}{\partial y^{\nu'}} B_\nu \quad . \end{aligned} \tag{3.17}$$

A particular example of a contraction is the scalar product between a vector and a covector which is a scalar.

Note that contraction over different pairs of indices will in general give rise to different tensors. E.g. $A^{\mu}_{\ \nu\mu}$ and $A^{\mu}_{\ \mu\nu}$ will in general be different.

4. Raising and Lowering of Indices

These operations can of course be combined in various ways. A particular important operation is, given a metric tensor, the raising and lowering of indices with the metric. From the above we know that given a (p, q)-tensor $A^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}$, the product plus contraction with the metric tensor $g_{\mu_1\nu}A^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}$ is a $(p-1, q+1)$-tensor. It will be denoted by the same symbol, but with *one index lowered by the metric*, i.e. we write

$$g_{\mu_1\nu}A^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} \equiv A_{\nu}{}^{\mu_2\ldots\mu_p}{}_{\nu_1\ldots\nu_q} \quad . \tag{3.18}$$

Note that there are p different ways of lowering the indices, and they will in general give rise to different tensors. It is therefore important to keep track of this in the notation. Thus, in the above, had we contracted over the second index instead of the first, we should write

$$g_{\mu_2\nu}A^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} \equiv A^{\mu_1}{}_{\nu}{}^{\mu_3\ldots\mu_p}{}_{\nu_1\ldots\nu_q} \quad . \tag{3.19}$$

Finally note that this notation is consistent with denoting the inverse metric by raised indices because

$$g^{\mu\nu} = g^{\mu\lambda}g^{\nu\sigma}g_{\lambda\sigma} \quad . \tag{3.20}$$

and raising one index of the metric gives the Kronecker tensor,

$$g^{\mu\lambda}g_{\lambda\nu} \equiv g^{\mu}_{\ \nu} = \delta^{\mu}_{\ \nu} \quad . \tag{3.21}$$

An observation we will frequently make use of to recognise when some object is a tensor is the following (occasionally known as the *quotient theorem* or quotient lemma):

Assume that you are given some object $A^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}$. Then if for every covector U_μ the contracted object $U_{\mu_1}A^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}$ transforms like a $(p-1, q)$-tensor, $A^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}$ is a (p, q)-tensor. Likewise for contractions with vectors or other tensors so that if e.g. in an equation of the form

$$A_{\mu\nu} = B_{\mu\nu\lambda\rho}C^{\lambda\rho} \tag{3.22}$$

you know that A transforms as a tensor for every tensor C, then B itself has to be a tensor.

3.4 Tensor Densities

While tensors are the objects which, in a sense, transform in the nicest and simplest possible way under coordinate transformations, they are not the only relevant objects.

An important class of non-tensors are so-called *tensor densities.* The prime example of a tensor density is the determinant $g := -\det g_{\mu\nu}$ of the metric tensor (the minus sign is there only to make g positive in signature $(-+++)$). It follows from the standard tensorial transformation law of the metric that under a coordinate transformation $x^\mu \rightarrow y^{\mu'}(x^\mu)$ this determinant transforms as

$$g' = \det\left(\frac{\partial x}{\partial y}\right)^2 g = \det\left(\frac{\partial y}{\partial x}\right)^{-2} g \quad . \tag{3.23}$$

An object which transforms in such a way under coordinate transformations is called a *scalar tensor density of weight* (-2). In general, a tensor density of weight w is an object that transforms as

$$T^{\mu'_1\ldots\mu'_p}_{\nu'_1\ldots\nu'_q} = \det\left(\frac{\partial y}{\partial x}\right)^w \frac{\partial y^{\mu'_1}}{\partial x^{\mu_1}}\cdots\frac{\partial y^{\mu'_p}}{\partial x^{\mu_p}}\frac{\partial x^{\nu_1}}{\partial y^{\nu'_1}}\cdots\frac{\partial x^{\nu_q}}{\partial y^{\nu'_q}}T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} \quad . \tag{3.24}$$

In particular, this implies that $g^{w/2}T^{\cdots}_{\cdots}$ transforms as (and hence is) a tensor,

$$g'^{w/2}T^{\mu'_1\ldots\mu'_p}_{\nu'_1\ldots\nu'_q} = \frac{\partial y^{\mu'_1}}{\partial x^{\mu_1}}\cdots\frac{\partial y^{\mu'_p}}{\partial x^{\mu_p}}\frac{\partial x^{\nu_1}}{\partial y^{\nu'_1}}\cdots\frac{\partial x^{\nu_q}}{\partial y^{\nu'_q}}g^{w/2}T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} \quad . \tag{3.25}$$

Conversely, therefore, any tensor density of weight w can be written as tensor times $g^{-w/2}$.

The relevance of tensor densities arises from the fundamental theorem of integral calculus that says that the integral measure d^4x (more generally d^nx in dimension n) transforms as

$$d^4y = \det\left(\frac{\partial y}{\partial x}\right) d^4x \quad , \tag{3.26}$$

i.e. as a scalar density of weight (+1). Thus $g^{1/2}d^4x$ is a volume element which is invariant under coordinate transformations and can be used to define integrals of scalars (functions) in a general metric (curved) space in a coordinate-independent way as

$$\int f := \int \sqrt{g} d^4x f(x) \quad . \tag{3.27}$$

This will of course be important in order to formulate action principles etc. in a metric space in a generally covariant way.

This is also frequently the quickest way to determine the volume element in non-Cartesian coordinates in Euclidean space. Thus, to determine what is the volume element in spherical coordinates $\{y^k\} = (r, \theta, \phi)$, say, instead of laboriously determining the Jacobi matrix for the coordinate transformation, and then (equally laboriously) calculating its determinant (which would be the standard uninspiring and uninspired procedure), all one needs to know is the metric in these coordinates to deduce

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2 d\phi^2) \quad \Rightarrow \quad g = r^4 \sin^2\theta \tag{3.28}$$

and therefore

$$d^3x = \sqrt{g}\, d^3y = r^2 \sin\theta \; dr \; d\theta \; d\phi \quad . \tag{3.29}$$

Note that all this is as it should be. In order to *measure* things like distances, areas and volumes, all that one should need to know is the *metric*, not the Jacobian between two coordinate systems.

There is one more important tensor density which - like the Kronecker tensor - has the same components in all coordinate systems. This is the totally anti-symmetric Levi-Civita symbol $\epsilon^{\mu\nu\rho\sigma}$ (taking the values $0, \pm 1$) which, as you can check, is a tensor density of weight (-1) so that $g^{-1/2}\epsilon^{\mu\nu\rho\sigma}$ is a tensor (strictly speaking it is a pseudo-tensor because of its behaviour under reversal of orientation but this will not concern us here).

The algebraic rules for tensor densities are strictly analogous to those for tensors. Thus, for example, the sum of two (p, q) tensor densities of weight w (let us call this a $(p, q; w)$ tensor) is again a $(p, q; w)$ tensor, and the direct product of a $(p_1, q_1; w_1)$ and a $(p_2, q_2; w_2)$ tensor is a $(p_1 + p_2, q_1 + q_2; w_1 + w_2)$ tensor. Contractions and the raising and lowering of indices of tensor densities can also be defined just as for ordinary tensors.

3.5 * A Coordinate-Independent Interpretation of Tensors

There is a more invariant and coordinate-independent way of looking at tensors than we have developed so far. The purpose of this section is to explain this point of view even though it is not indispensable for an understanding of the remainder of the course.

Consider first of all the derivative *df* of a function (scalar field) $f = f(x)$. This is clearly a coordinate-independent object, not only because we didn't have to specify a coordinate system to write *df* but also because

$$df = \frac{\partial f(x)}{\partial x^\mu} dx^\mu = \frac{\partial f(y(x))}{\partial y^{\mu'}} dy^{\mu'} \quad , \tag{3.30}$$

which follows from the fact that $\partial_\mu f$ (a covector) and dx^μ (the coordinate differentials) transform inversely to each other under coordinate transformations. This suggests that it is useful to regard the quantities $\partial_\mu f$ as the coefficients of the coordinate independent object *df* in a particular coordinate system, namely when *df* is expanded in the basis $\{dx^\mu\}$.

We can do the same thing for any covector U_μ. If U_μ is a covector (i.e. transforms like one under coordinate transformations), then $U := U_\mu(x) dx^\mu$ is coordinate-independent, and it is useful to think of the U_μ as the coefficients of the covector U when expanded in a coordinate basis, $U = U_\mu dx^\mu$.

We can even do the same thing for a general covariant tensor $T_{\mu\nu\cdots}$. Namely, if $T_{\mu_1\cdots\mu_q}$ is a $(0,q)$-tensor, then

$$T := T_{\mu_1\cdots\mu_q} dx^{\mu_1} \ldots dx^{\mu_q} \tag{3.31}$$

is coordinate independent. In the particular case of the metric tensor we have already known and used this. In that case, T is what we called ds^2, $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, which we know to be invariant under coordinate transformations.

Now, can we do something similar for vectors and other contravariant (or mixed) tensors? The answer is yes. Just as covectors transform inversely to coordinate differentials, vectors V^μ transform inversely to partial derivatives ∂_μ. Thus

$$V := V^\mu(x) \frac{\partial}{\partial x^\mu} \tag{3.32}$$

is coordinate dependent - a coordinate-independent linear first-order differential operator. One can thus always think of a vector field as a differential operator and this is a very fruitful point of view.

Acting on a function (scalar) f, V produces the derivative of f along V,

$$Vf = V^\mu \partial_\mu f \ . \tag{3.33}$$

This is also a coordinate independent object, a scalar, arising from the contraction of a vector and a covector. And this is as it should be because, after all, both a function and a vector field can be specified on a space-time without having to introduce coordinates (e.g. by simply drawing the vector field and the profile of the function). Therefore also the change of the function along a vector field should be coordinate independent and, as we have seen, it is.

Also this can, in principle, be extended to higher rank tensors, but at this point it would be very useful to introduce the notion of tensor product, something I will not do. Fact of the matter is, however, that any (p,q)-tensor $T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}$ can be thought of as the collection of components of a coordinate independent object T when expanded in a particular coordinate basis in terms of the dx^μ and $(\partial/\partial x^\mu)$.

4 Tensor Analysis

Tensors transform in a nice and simple way under general coordinate transformations. Thus these appear to be the right objects to construct equations from that satisfy the Principle of General Covariance.

However, the laws of physics are differential equations, so we need to know how to differentiate tensors. The problem is that the ordinary partial derivative does not map tensors to tensors, the partial derivative of a (p,q)-tensor is not a tensor unless $p=q=0$.

This is easy to see: take for example a vector V^μ. Under a coordinate transformation, its partial derivative transforms as

$$\begin{aligned}\partial_{\nu'}V^{\mu'} &= \frac{\partial x^\nu}{\partial y^{\nu'}}\frac{\partial}{\partial x^\nu}\frac{\partial y^{\mu'}}{\partial x^\mu}V^\mu \\ &= \frac{\partial x^\nu}{\partial y^{\nu'}}\frac{\partial y^{\mu'}}{\partial x^\mu}\partial_\nu V^\mu + \frac{\partial x^\nu}{\partial y^{\nu'}}\frac{\partial^2 y^{\mu'}}{\partial x^\mu \partial x^\nu}V^\mu \quad .\end{aligned} \tag{4.1}$$

The appearance of the second term shows that the partial derivative of a vector is not a tensor.

As the second term is zero for linear transformations, you see that partial derivatives transform in a tensorial way e.g. under Lorentz transformations, so that partial derivatives are all one usually needs in special relativity.

We also see that the lack of covariance of the partial derivative is very similar to the lack of covariance of the equation $\ddot{x}^\mu = 0$, and this suggests that the problem can be cured in the same way - by introducing Christoffel symbols. This is indeed the case.

4.1 The Covariant Derivative for Vector Fields

Let us define the *covariant derivative* $\nabla_\nu V^\mu$ of a vector field V^μ by

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\lambda}V^\lambda \quad . \tag{4.2}$$

It follows from the non-tensorial behaviour (1.67) of the Christoffel symbols under coordinate transformations that $\nabla_\nu V^\mu$, as defined above, is indeed a $(1,1)$ tensor. Moreover, in a locally inertial coordinate system this reduces to the ordinary partial derivative, and we have thus, as desired, arrived at an appropriate tensorial generalisation of the partial derivative operator.

We could have also arrived at the above definition in a somewhat more systematic way. Namely, let $\{\xi^A\}$ be an inertial coordinate system. In an inertial coordinate system we can just use the ordinary partial derivative $\partial_B V^A$. We now define the new (improved, covariant) derivative $\nabla_\nu V^\mu$ in any other coordinate system $\{x^\mu\}$ by demanding that it transforms as a (1,1)-tensor, i.e. we define

$$\nabla_\nu V^\mu := \frac{\partial x^\mu}{\partial \xi^A}\frac{\partial \xi^B}{\partial x^\nu}\partial_B V^A \quad . \tag{4.3}$$

By a straightforward calculation one finds that

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\ \nu\lambda} V^\lambda \quad , \tag{4.4}$$

where $\Gamma^\mu_{\ \nu\lambda}$ is our old friend

$$\Gamma^\mu_{\ \nu\lambda} = \frac{\partial x^\mu}{\partial \xi^A}\frac{\partial^2 \xi^A}{\partial x^\nu \partial x^\lambda} \quad . \tag{4.5}$$

We can thus adopt (4.4) as our definition of the *covariant derivative* in a general metric space or space-time (with the Christoffel symbols calculated from the metric in the usual way).

That $\nabla_\mu V^\nu$, defined in this way, is indeed a $(1,1)$ tensor, now follows directly from the way we arrived at the definition of the covariant derivative. Indeed, imagine transforming from inertial coordinates to another coordinate system $\{y^{\mu'}\}$. Then (4.3) is replaced by

$$\nabla_{\nu'} V^{\mu'} := \frac{\partial y^{\mu'}}{\partial \xi^A}\frac{\partial \xi^B}{\partial y^{\nu'}}\partial_B V^A \quad . \tag{4.6}$$

Comparing this with (4.3), we see that the two are related by

$$\nabla_{\nu'} V^{\mu'} := \frac{\partial y^{\mu'}}{\partial x^\mu}\frac{\partial x^\nu}{\partial y^{\nu'}}\nabla_\nu V^\mu \quad , \tag{4.7}$$

as required.

A word on notation: Frequently, the covariant derivative $\nabla_\nu V^\mu$ is also denoted by a semicolon, $\nabla_\nu V^\mu = V^\mu;_\nu$. Just as for functions, one can also define the covariant directional derivative of a vector field V along another vector field X^μ by

$$\nabla_X V^\mu \equiv X^\nu \nabla_\nu V^\mu \quad . \tag{4.8}$$

4.2 * Invariant Interpretation of the Covariant Derivative

The appearance of the Christoffel-term in the definition of the covariant derivative may at first sight appear a bit unusual (even though it also appears when one just transforms Cartesian partial derivatives to polar coordinates etc.). There is a more invariant way of explaining the appearance of this term, related to the more coordinate-independent way of looking at tensors explained above. Namely, since the $V^\mu(x)$ are really just the coefficients of the vector field $V(x) = V^\mu(x)\partial_\mu$ when expanded in the basis ∂_μ, a meanigful definition of the derivative of a vector field must take into account not only the change in the coefficients but also the fact that the basis changes from point to point - and this is precisely what the Christoffel symbols do. Writing

$$\begin{aligned}\nabla_\nu V &= \nabla_\nu(V^\mu \partial_\mu) \\ &= (\partial_\nu V^\mu)\partial_\mu + V^\lambda(\nabla_\nu \partial_\lambda) \ ,\end{aligned} \tag{4.9}$$

we see that we reproduce the definition of the covariant derivative if we set

$$\nabla_\nu \partial_\lambda = \Gamma^\mu_{\ \nu\lambda}\partial_\mu \ . \tag{4.10}$$

Indeed we then have

$$\nabla_\nu V \equiv (\nabla_\nu V^\mu)\partial_\mu = (\partial_\nu V^\mu + \Gamma^\mu_{\ \nu\lambda}V^\lambda)\partial_\mu \ , \tag{4.11}$$

which agrees with the above definition.

It is instructive to check in some examples that the Christoffel symbols indeed describe the change of the tangent vectors ∂_μ. For instance on the plane, in polar coordinates, one has

$$\nabla_r \partial_r = \Gamma^\mu_{\ rr}\partial_\mu = 0 \ , \tag{4.12}$$

which is correct because ∂_r indeed does not change when one moves in the radial direction. ∂_r does change, however, when one moves in the angular direction given by ∂_ϕ. In fact, it changes its direction proportional to ∂_ϕ but this change is stronger for small values of r than for larger ones (draw a picture!). This is precisely captured by the non-zero Christoffel symbol $\Gamma^\phi_{\ r\phi}$,

$$\nabla_\phi \partial_r = \Gamma^\phi_{\ r\phi}\partial_\phi = \frac{1}{r}\partial_\phi \ . \tag{4.13}$$

4.3 Extension of the Covariant Derivative to Other Tensor Fields

Our basic postulates for the covariant derivative are the following:

1. Linearity and Tensoriality

   ∇_μ is a linear operator that maps (p,q)-tensors to $(p,q+1)$-tensors

2. Generalisation of the Partial Derivative

   On scalars ϕ, the covariant derivative ∇_μ reduces to the ordinary partial derivative (since $\partial_\mu\phi$ is already a covector),

$$\nabla_\mu \phi = \partial_\mu \phi \ . \tag{4.14}$$

3. Leibniz Rule

   Acting on the direct product of tensors, ∇_μ satisfies a generalised Leibniz rule,

$$\nabla_\mu(A^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}B^{\rho_1\ldots\rho_r}_{\lambda_1\ldots\lambda_s}) = \nabla_\mu(A^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q})B^{\rho_1\ldots\rho_r}_{\lambda_1\ldots\lambda_s} + A^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q}\nabla_\mu B^{\rho_1\ldots\rho_r}_{\lambda_1\ldots\lambda_s} \tag{4.15}$$

We will now see that, demanding the above properties, in particular the Leibniz rule, there is a unique extension of the covariant derivative on vector fields to a differential operator on general tensor fields, mapping (p,q)- to $(p,q+1)$-tensors.

To define the covariant derivative for covectors U_μ, we note that $U_\mu V^\mu$ is a scalar for any vector V^μ so that

$$\nabla_\mu(U_\nu V^\nu) = \partial_\mu(U_\nu V^\nu) = (\partial_\mu U_\nu)V^\nu + U_\nu(\partial_\mu V^\nu) \tag{4.16}$$

(since the partial derivative satisfies the Leibniz rule), and we demand

$$\nabla_\mu(U_\nu V^\nu) = (\nabla_\mu U_\nu)V^\nu + U_\nu \nabla_\mu V^\nu \ \ . \tag{4.17}$$

As we know $\nabla_\mu V^\nu$, these two equations determine $\nabla_\mu U_\nu$ uniquely to be

$$\nabla_\mu U_\nu = \partial_\mu U_\nu - \Gamma^\lambda_{\mu\nu} U_\lambda \ \ . \tag{4.18}$$

That this is indeed a $(0,2)$-tensor can either be checked directly or, alternatively, is a consequence of the quotient theorem.

The extension to other (p,q)-tensors is now immediate. If the (p,q)-tensor is the direct product of p vectors and q covectors, then we already know its covariant derivative (using the Leibniz rule again). We simply adopt the same resulting formula for an arbitrary (p,q)-tensor. The result is that the covariant derivative of a general (p,q)-tensor is the sum of the partial derivative, a Christoffel symbol with a positive sign for each of the p upper indices, and a Christoffel with a negative sign for each of the q lower indices. In equations

$$\begin{aligned} \nabla_\mu T^{\nu_1\cdots\nu_p}{}_{\rho_1\cdots\rho_q} &= \partial_\mu T^{\nu_1\cdots\nu_p}{}_{\rho_1\cdots\rho_q} \\ &+ \underbrace{\Gamma^{\nu_1}_{\mu\lambda} T^{\lambda\nu_2\cdots\nu_p}{}_{\rho_1\cdots\rho_q} + \ldots + \Gamma^{\nu_p}_{\mu\lambda} T^{\nu_1\cdots\nu_{p-1}\lambda}{}_{\rho_1\cdots\rho_q}}_{p \text{ terms}} \\ &- \underbrace{\Gamma^{\lambda}_{\mu\rho_1} T^{\nu_1\cdots\nu_p}{}_{\lambda\rho_2\cdots\rho_q} - \ldots - \Gamma^{\lambda}_{\mu\rho_q} T^{\nu_1\cdots\nu_p}{}_{\rho_1\cdots\rho_{q-1}\lambda}}_{q \text{ terms}} \end{aligned} \tag{4.19}$$

Having defined the covariant derivative for arbitrary tensors, we are also ready to define it for tensor densities. For this we recall that if T is a $(p,q;w)$ tensor density, then $g^{w/2}T$ is a (p,q)-tensor. Thus $\nabla_\mu(g^{w/2}T)$ is a $(p,q+1)$-tensor. To map this back to a tensor density of weight w, we multiply this by $g^{-w/2}$, arriving at the definition

$$\nabla_\mu T := g^{-w/2}\nabla_\mu(g^{w/2}T) \ \ . \tag{4.20}$$

Working this out explictly, one finds

$$\nabla_\mu T = \frac{w}{2g}(\partial_\mu g)T + \nabla^{\text{tensor}}_\mu T \ \ , \tag{4.21}$$

where $\nabla_\mu^{\text{tensor}}$ just means the usual covariant derivative for (p,q)-tensors defined above. For example, for a scalar density ϕ one has

$$\nabla_\mu\phi = \partial_\mu\phi + \frac{w}{2g}(\partial_\mu g)\phi \quad . \tag{4.22}$$

In particular, since the determinant g is a scalar density of weight -2, it follows that

$$\nabla_\mu g = 0 \quad , \tag{4.23}$$

which obviously simplifies integrations by parts in integrals defined with the measure $\sqrt{g}d^4x$.

4.4 Main Properties of the Covariant Derivative

The main properties of the covariant derivative, in addition to those that were part of our postulates (like linearity and the Leibniz rule) are the following:

1. ∇_μ Commutes with Contraction

   This means that if A is a (p,q)-tensor and B is the $(p-1,q-1)$-tensor obtained by contraction over two particular indices, then the covariant derivative of B is the same as the covariant derivative of A followed by contraction over these two indices. This comes about because of a cancellation between the corresponding two Christoffel symbols with opposite signs. Consider e.g. a (1,1)-tensor $A^\nu{}_\rho$ and its contraction $A^\nu{}_\nu$. The latter is a scalar and hence its covariant derivative is just the partial derivative. This can also be obtained by taking first the covariant derivative of A,

$$\nabla_\mu A^\nu{}_\rho = \partial_\mu A^\nu{}_\rho + \Gamma^\nu{}_{\mu\lambda}A^\lambda{}_\rho - \Gamma^\lambda{}_{\mu\rho}A^\nu{}_\lambda \quad , \tag{4.24}$$

   and then contracting:

$$\nabla_\mu A^\nu{}_\nu = \partial_\mu A^\nu{}_\nu + \Gamma^\nu{}_{\mu\lambda}A^\lambda{}_\nu - \Gamma^\lambda{}_{\mu\nu}A^\nu{}_\lambda = \partial_\mu A^\nu{}_\nu \quad . \tag{4.25}$$

   The most transparent way of stating this property is that the Kronecker delta is covariantly constant, i.e. that

$$\nabla_\mu \delta^\nu{}_\lambda = 0 \quad . \tag{4.26}$$

   To see this, we use the Leibniz rule to calculate

$$\begin{aligned}
\nabla_\mu A^{\nu\dots}_{\nu\dots} &= \nabla_\mu(A^{\nu\dots}_{\rho\dots}\delta^\rho{}_\nu) \\
&= (\nabla_\mu A^{\nu\dots}_{\rho\dots})\delta^\rho{}_\nu + A^{\nu\dots}_{\rho\dots}\nabla_\mu\delta^\rho{}_\nu \\
&= (\nabla_\mu A^{\nu\dots}_{\rho\dots})\delta^\rho{}_\nu
\end{aligned} \tag{4.27}$$

which is precisely the statement that covariant differentiation and contraction commute. To establish that the Kronecker delta is covariantly constant, we follow the rules to find

$$\begin{aligned}\nabla_\mu \delta^\nu_{\ \lambda} &= \partial_\mu \delta^\nu_{\ \lambda} + \Gamma^\nu_{\ \mu\rho}\delta^\rho_{\ \lambda} - \Gamma^\rho_{\ \mu\lambda}\delta^\nu_{\ \rho} \\ &= \Gamma^\nu_{\ \mu\lambda} - \Gamma^\nu_{\ \mu\lambda} = 0 \ .\end{aligned} \tag{4.28}$$

2. The Metric is Covariantly Constant: $\nabla_\mu g_{\nu\lambda} = 0$

   This is one of the key properties of the covariant derivative ∇_μ we have defined. I will give two arguments to establish this:

   (a) Since $\nabla_\mu g_{\nu\lambda}$ is a tensor, we can choose any coordinate system we like to establish if this tensor is zero or not at a given point x. Choose an inertial coordinate system at x. Then the partial derivatives of the metric and the Christoffel symbols are zero there. Therefore the covariant derivative of the metric is zero. Since $\nabla_\mu g_{\nu\lambda}$ is a tensor, this is then true in every coordinate system.

   (b) The other argument is by direct calculation. Recalling the identity

   $$\partial_\mu g_{\nu\lambda} = \Gamma_{\nu\lambda\mu} + \Gamma_{\lambda\nu\mu} \ , \tag{4.29}$$

   we calculate

   $$\begin{aligned}\nabla_\mu g_{\nu\lambda} &= \partial_\mu g_{\nu\lambda} - \Gamma^\rho_{\ \mu\nu} g_{\rho\lambda} - \Gamma^\rho_{\ \mu\lambda} g_{\nu\rho} \\ &= \Gamma_{\nu\lambda\mu} + \Gamma_{\lambda\nu\mu} - \Gamma_{\lambda\mu\nu} - \Gamma_{\nu\mu\lambda} \\ &= 0 \ .\end{aligned} \tag{4.30}$$

3. ∇_μ Commutes with Raising and Lowering of Indices

   This is really a direct consequence of the covariant constancy of the metric. For example, if V_μ is the covector obtained by lowering an index of the vector V^μ, $V_\mu = g_{\mu\nu}V^\nu$, then

   $$\nabla_\lambda V_\mu = \nabla_\lambda (g_{\mu\nu} V^\nu) = g_{\mu\nu}\nabla_\lambda V^\nu \ . \tag{4.31}$$

4. Covariant Derivatives Commute on Scalars

   This is of course a familiar property of the ordinary partial derivative, but it is also true for the second covariant derivatives of a scalar and is a consequence of the symmetry of the Christoffel symbols in the second and third indices and is also knowns as the *no torsion* property of the covariant derivative. Namely, we have

   $$\begin{aligned}\nabla_\mu\nabla_\nu\phi - \nabla_\nu\nabla_\mu\phi &= \nabla_\mu\partial_\nu\phi - \nabla_\nu\partial_\mu\phi \\ &= \partial_\mu\partial_\nu\phi - \Gamma^\lambda_{\ \mu\nu}\phi - \partial_\nu\partial_\mu\phi + \Gamma^\lambda_{\ \nu\mu}\phi = 0 \ .\end{aligned} \tag{4.32}$$

   Note that *the second covariant derivatives on higher rank tensors do not commute* - we will come back to this in our discussion of the curvature tensor later on.

4.5 Tensor Analysis: Some Special Cases

In this section I will, without proof, give some useful special cases of covariant derivatives - covariant curl and divergence etc. - you should make sure that you can derive all of these yourself without any problems.

1. The Covariant Curl of a Covector

   One has
   $$\nabla_\mu U_\nu - \nabla_\nu U_\mu = \partial_\mu U_\nu - \partial_\nu U_\mu \ \ , \tag{4.33}$$
   because the symmetric Christoffel symbols drop out in this antisymmetric linear combination. Thus the Maxwell field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is a tensor under general coordinate transformations, no metric or covariant derivative is needed to make it a tensor in a general space time.

2. The Covariant Curl of an Antisymmetric Tensor

   Let $A_{\nu\lambda\ldots}$ be completely antisymmetric. Then, as for the curl of covectors, the metric and Christoffel symbols drop out of the expression for the curl, i.e. one has
   $$\nabla_{[\mu} A_{\nu\lambda\cdots]} = \partial_{[\mu} A_{\nu\lambda\cdots]} \ \ . \tag{4.34}$$
   Here the square brackets on the indices denote complete antisymmetrisation. In particular, the Bianchi identity for the Maxwell field strength tensor is independent of the metric also in a general metric space time.

3. The Covariant Divergence of a Vector

   By the covariant divergence of a vector field one means the scalar
   $$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\ \mu\lambda} V^\lambda \ \ . \tag{4.35}$$
   Now a useful identity for the contracted Christoffel symbol is
   $$\Gamma^\mu_{\ \mu\lambda} = g^{-1/2}\partial_\lambda (g^{+1/2}) \ \ . \tag{4.36}$$
   Here is an elementary proof of this identity (an alternative standard proof can be based on the identity $\det M = \exp \operatorname{tr} \log M$ and its derivative or variation): the standard (cofactor or minor) expansion formula for the determinant is
   $$g = \sum_\nu (-1)^{\mu+\nu} g_{\mu\nu} |m_{\mu\nu}| \ \ , \tag{4.37}$$
   where $|m_{\mu\nu}|$ is the determinant of the minor of $g_{\mu\nu}$, i.e. of the matrix one obtains by removing the μ'th row and ν'th column from $g_{\mu\nu}$. It follows that
   $$\frac{\partial g}{\partial g_{\mu\nu}} = (-1)^{\mu+\nu} |m_{\mu\nu}| \ \ . \tag{4.38}$$

Another consequence of (4.37) is

$$\sum_\nu (-1)^{\mu+\nu} g_{\lambda\nu}|m_{\mu\nu}| = 0 \qquad \lambda \neq \mu \ , \tag{4.39}$$

since this is, in particular, the determinant of a matrix with $g_{\mu\nu} = g_{\lambda\nu}$, i.e. of a matrix with two equal rows.

Together, these two results can be written as

$$\sum_\nu (-1)^{\mu+\nu} g_{\lambda\nu}|m_{\mu\nu}| = \delta_{\lambda\mu} g \ . \tag{4.40}$$

Multiplying (4.38) by $g_{\lambda\nu}$ and using (4.40), one finds

$$g_{\lambda\nu}\frac{\partial g}{\partial g_{\mu\nu}} = \delta_{\lambda\mu} g \tag{4.41}$$

or the simple identity

$$\frac{\partial g}{\partial g_{\mu\nu}} = g^{\mu\nu} g \ . \tag{4.42}$$

Thus

$$\partial_\lambda g = \frac{\partial g}{\partial g_{\mu\nu}}\partial_\lambda g_{\mu\nu} = g g^{\mu\nu}\partial_\lambda g_{\mu\nu} \ . \tag{4.43}$$

or

$$g^{-1}\partial_\lambda g = g^{\mu\nu}\partial_\lambda g_{\mu\nu} \ . \tag{4.44}$$

On the other hand, the contracted Christoffel symbol is

$$\Gamma^\mu_{\ \mu\lambda} = \frac{1}{2} g^{\mu\nu}\partial_\lambda g_{\mu\nu} \ , \tag{4.45}$$

which establishes the identity (4.36).

Thus the covariant divergence can be written compactly as

$$\nabla_\mu V^\mu = g^{-1/2}\partial_\mu (g^{+1/2} V^\mu) \ , \tag{4.46}$$

and one only needs to calculate g and its derivative, not the Christoffel symbols themselves, to calculate the covariant divergence of a vector field.

4. The Covariant Laplacian of a Scalar

How should the Laplacian be defined? Well, the obvious guess (something that is covariant and reduces to the ordinary Laplacian for the Minkowski metric) is $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$, which can alternatively be written as

$$\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu = \nabla^\mu\nabla_\mu = \nabla_\mu\nabla^\mu = \nabla_\mu g^{\mu\nu}\nabla_\nu \tag{4.47}$$

etc. Note that, even though the covariant derivative on scalars reduces to the ordinary partial derivative, so that one can write

$$\Box\phi = \nabla_\mu g^{\mu\nu}\partial_\nu\phi \ , \tag{4.48}$$

it makes no sense to write this as $\nabla_\mu\partial^\mu\phi$: since ∂_μ does not commute with the metric in general, the notation ∂^μ is at best ambiguous as it is not clear whether this should represent $g^{\mu\nu}\partial_\nu$ or $\partial_\nu g^{\mu\nu}$ or something altogether different. This ambiguity does not arise for the Minkowski metric, but of course it is present in general.

A compact yet explicit expression for the Laplacian follows from the expression for the covariant divergence of a vector:

$$\begin{aligned}\Box\phi &:= g^{\mu\nu}\nabla_\mu\nabla_\nu\phi \\ &= \nabla_\mu(g^{\mu\nu}\partial_\nu\phi) \\ &= g^{-1/2}\partial_\mu(g^{1/2}g^{\mu\nu}\partial_\nu\phi) \ .\end{aligned} \tag{4.49}$$

This formula is also useful (and provides the quickest way of arriving at the result) if one just wants to write the ordinary flat space Laplacian on $\mathbb{R}^3$ in, say, polar or cylindrical coordinates.

To illustrate this, let us calculate the Laplacian for the standard metric on $\mathbb{R}^{n+1}$ in polar coordinates. The standard procedure would be to first determine the coordinate transformation $x^i = x^i(r, \text{angles})$, then calculate $\partial/\partial x^i$, and finally assemble all the bits and pieces to calculate $\Delta = \sum_i(\partial/\partial x^i)^2$. This is a pain.

To calculate the Laplacian, we do not need to know the coordinate transformation, all we need is the metric. In polar coordinates, this metric takes the form

$$ds^2(\mathbb{R}^{n+1}) = dr^2 + r^2 d\Omega_n^2 \ , \tag{4.50}$$

where $d\Omega_n^2$ is the standard line-element on the unit n-sphere S^n. The determinant of this metric is $g \sim r^{2n}$ (times a function of the coordinates (angles) on the sphere). Thus, for $n = 1$ one has $ds^2 = dr^2 + r^2 d\phi^2$ and therefore

$$\Delta = r^{-1}\partial_\mu(rg^{\mu\nu}\partial_\nu) = r^{-1}\partial_r(r\partial_r) + r^{-2}\partial_\phi^2 = \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\phi^2 \ . \tag{4.51}$$

In general, denoting the angular part of the Laplacian, i.e. the Laplacian of S^n, by Δ_{S^n}, one finds analogously

$$\Delta = \partial_r^2 + nr^{-1}\partial_r + r^{-2}\Delta_{S^n} \ . \tag{4.52}$$

I hope you agree that this method is superior to the standard procedure.

5. The Covariant Form of Gauss' Theorem

Let V^μ be a vector field, $\nabla_\mu V^\mu$ its divergence and recall that integrals in curved space are defined with respect to the integration measure $g^{1/2}d^4x$. Thus one has

$$\int g^{1/2}d^4x\nabla_\mu V^\mu = \int d^4x\partial_\mu(g^{1/2}V^\mu) \ . \tag{4.53}$$

Now the second term is an ordinary total derivative and thus, if V^μ vanishes sufficiently rapidly at infinity, one has

$$\int g^{1/2}d^4x\nabla_\mu V^\mu = 0 \ . \tag{4.54}$$

6. The Covariant Divergence of an Antisymmetric Tensor

   For a $(p,0)$-tensor $T^{\mu\nu\cdots}$ one has

$$\begin{aligned}\nabla_\mu T^{\mu\nu\cdots} &= \partial_\mu T^{\mu\nu\cdots} + \Gamma^\mu_{\mu\lambda}T^{\lambda\nu\cdots} + \Gamma^\nu_{\mu\lambda}T^{\mu\lambda\cdots} + \ldots \\ &= g^{-1/2}\partial_\mu(g^{1/2}T^{\mu\nu\cdots}) + \Gamma^\nu_{\mu\lambda}T^{\mu\lambda\cdots} + \ldots \quad . \end{aligned} \tag{4.55}$$

   In particular, if $T^{\mu\nu\cdots}$ is completely antisymmetric, the Christoffel terms disappear and one is left with

$$\nabla_\mu T^{\mu\nu\cdots} = g^{-1/2}\partial_\mu(g^{1/2}T^{\mu\nu\cdots}) \ . \tag{4.56}$$

7. The Lie derivative of the Metric

   In section 2.5 we had encountered the expression (2.62) for the variation of the metric under an infinitesimal coordinate transformation $\delta x^\alpha = \epsilon V^\alpha$,

$$\delta_V g_{\alpha\beta} = V^\gamma\partial_\gamma g_{\alpha\beta} + (\partial_\alpha V^\gamma)g_{\gamma\beta} + (\partial_\beta V^\gamma)g_{\alpha\gamma} \ . \tag{4.57}$$

   While we saw that this expression could be understood and deduced from the requirement that the variation of the metric is itself a tensorial object that transforms like the metric, the tensorial nature of the above expression is far from manifest. However, it has a very nice and simple expression in terms of covariant derivatives of V, namely

$$\delta_V g_{\alpha\beta} = \nabla_\alpha V_\beta + \nabla_\beta V_\alpha \tag{4.58}$$

   (as is easily verified). This expression will be rederived (and placed into the general context of Lie derivatives and Killing vectors) in section 6.

You will have noticed that many equations simplify considerably for completely antisymmetric tensors. In particular, their curl can be defined in a tensorial way without reference to any metric. This observation is at the heart of the coordinate independent calculus of *differential forms*. In this context, the curl is known as the *exterior derivative*.

Likewise, Lie derivatives of tensors (section 6 and item 7 above) are automatically tensorial objects (and one can, but need not, make their tensorial nature manifest by writing these derivatives in terms of covariant derivatives).

4.6 Covariant Differentiation Along a Curve

So far, we have defined covariant differentiation for tensors defined everywhere in space time. Frequently, however, one encounters tensors that are only defined on curves - like the momentum of a particle which is only defined along its world line. In this section we will see how to define covariant differentiation along a curve. Thus consider a curve

$x^\mu(\tau)$ (where τ could be, but need not be, proper time) and the tangent vector field $X^\mu(x(\tau)) = \dot{x}^\mu(\tau)$. Now define the covariant derivative $D/D\tau$ along the curve by

$$\frac{D}{D\tau} = X^\mu \nabla_\mu = \dot{x}^\mu \nabla_\mu \quad . \tag{4.59}$$

For example, for a vector one has

$$\begin{aligned} \frac{DV^\mu}{D\tau} &= \dot{x}^\nu \partial_\nu V^\mu + \dot{x}^\nu \Gamma^\mu_{\ \nu\lambda} V^\lambda \\ &= \frac{d}{d\tau} V^\mu(x(t)) + \Gamma^\mu_{\ \nu\lambda}(x(t)) \dot{x}^\nu(t) V^\lambda(x(t)) \quad . \end{aligned} \tag{4.60}$$

For this to make sense, V^μ needs to be defined only along the curve and not necessarily everywhere in space time.

This notion of covariant derivative along a curve permits us, in particular, to define the (covariant) acceleration a^μ of a curve $x^\mu(\tau)$ as the covariant derivative of the velocity $u^\mu = \dot{x}^\mu$,

$$a^\mu = \frac{D}{D\tau} \dot{x}^\mu = \ddot{x}^\mu + \Gamma^\mu_{\ \nu\lambda} \dot{x}^\nu \dot{x}^\lambda = u^\nu \nabla_\nu u^\mu \quad . \tag{4.61}$$

Thus we can characterise (affinely parametrised) geodesics as those curves whose acceleration is zero, a reasonable and natural statement regarding the movement of freely falling particles. If they are not affinely parametrised, as in (2.25), then instead of $u^\nu \nabla_\nu u^\mu = 0$ one has

$$u^\nu \nabla_\nu u^\mu = \kappa u^\mu \quad . \tag{4.62}$$

4.7 Parallel Transport and Geodesics

We now come to the important notion of *parallel transport* of a tensor along a curve. Note that, in a general (curved) metric space time, it does not make sense to ask if two vectors defined at points x and y are parallel to each other or not. However, given a metric and a curve connecting these two points, one can compare the two by dragging one along the curve to the other using the covariant derivative.

We say that a tensor $T^{\cdots}_{\cdots}$ is *parallel transported along the curve* $x^\mu(\tau)$ if

$$\frac{DT^{\cdots}_{\cdots}}{D\tau} = 0 \quad . \tag{4.63}$$

Here are some immediate consequences of this definition:

1. In a locally inertial coordinate system along the curve, this condition reduces to $dT/d\tau = 0$, i.e. to the statement that the tensor does not change along the curve. Thus the above is indeed an appropriate tensorial generalisation of the intuitive notion of parallel transport to a general metric space-time.

2. The parallel transport condition is a first order differential equation along the curve and thus defines $T^{\dots}_{\dots}(\tau)$ given an initial value $T^{\dots}_{\dots}(\tau_0)$.

3. Taking T to be the tangent vector $X^\mu = \dot{x}^\mu$ to the curve itself, the condition for parallel transport becomes

$$\frac{D}{D\tau}X^\mu = 0 \Leftrightarrow \ddot{x}^\mu + \Gamma^\mu_{\ \nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0 \ , \tag{4.64}$$

i.e. precisely the geodesic equation. Thus geodesics, as we have already seen these are curves with zero acceleration, can equivalently be characterised by the property that their tangent vectors are parallel transported (do not change) along the curve. For this reason geodesics are also known as *autoparallels*.

4. Since the metric is covariantly constant, it is in particular parallel along any curve. Thus, in particular, if V^μ is parallel transported, also its length remains constant along the curve,

$$\frac{DV^\mu}{D\tau} = 0 \Rightarrow \frac{D}{D\tau}(g_{\mu\nu}V^\mu V^\nu) = 0 \ . \tag{4.65}$$

In particular, we rediscover the fact claimed in (2.17) that the quantity $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = g_{\mu\nu}X^\mu X^\nu$ is constant along a geodesic.

5. Now let $x^\mu(\tau)$ be a geodesic and V^μ parallel along this geodesic. Then, as one might intuitively expect, also the angle between V^μ and the tangent vector to the curve X^μ remains constant. This is a consequence of the fact that both the norm of V and the norm of X are constant along the curve and that

$$\begin{aligned} \frac{d}{d\tau}(g_{\mu\nu}X^\mu V^\nu) &= \frac{D}{D\tau}(g_{\mu\nu}X^\mu V^\nu) \\ &= g_{\mu\nu}\frac{D}{D\tau}(X^\mu)V^\nu + g_{\mu\nu}X^\mu\frac{D}{D\tau}V^\nu \\ &= 0 + 0 = 0 \end{aligned} \tag{4.66}$$

6. The physical meaning of parallel transport of a vector along a curve is that it corresponds to a physically invariant direction as determined e.g. by a Foucault pendulum or a perfect gyroscope.

4.8 * Generalisations

Recall that the transformation behaviour of the Christoffel symbols, equation (1.67), was the key ingredient in the proof that the geodesic equation transforms like a vector under general coordinate transformations. Likewise, to show that the covariant derivative of a tensor is again a tensor all one needs to know is that the Christoffel symbols transform in this way. Thus any other object $\tilde{\Gamma}^\mu_{\ \nu\lambda}$ could also be used to define a covariant derivative (generalizing the partial derivative and mapping tensors to tensors) provided that it transforms in the same way as the Christoffel symbols, i.e. provided that one has

$$\tilde{\Gamma}^{\mu'}_{\ \nu'\lambda'} = \tilde{\Gamma}^{\mu}_{\ \nu\lambda}\frac{\partial y^{\mu'}}{\partial x^{\mu}}\frac{\partial x^{\nu}}{\partial y^{\nu'}}\frac{\partial x^{\lambda}}{\partial y^{\lambda'}} + \frac{\partial y^{\mu'}}{\partial x^{\mu}}\frac{\partial^2 x^{\mu}}{\partial y^{\nu'}\partial y^{\lambda'}} \quad . \tag{4.67}$$

But this implies that the difference

$$\Delta^{\mu}_{\ \nu\lambda} = \tilde{\Gamma}^{\mu}_{\ \nu\lambda} - \Gamma^{\mu}_{\ \nu\lambda} \tag{4.68}$$

transforms as a tensor. Thus, any such $\tilde{\Gamma}$ is of the form

$$\tilde{\Gamma}^{\mu}_{\ \nu\lambda} = \Gamma^{\mu}_{\ \nu\lambda} + \Delta^{\mu}_{\ \nu\lambda} \tag{4.69}$$

where Δ is a (1,2)-tensor and the question arises if or why the Christoffel symbols we have been using are somehow singled out or preferred.

In some sense, the answer is an immediate yes because it is this particular connection that enters in determining the paths of freely falling particles (the geodesics which extremise proper time).

Moreover, the covariant derivative as we have defined it has two important properties, namely

1. that the metric is covariantly constant, $\nabla_\mu g_{\nu\lambda} = 0$, and
2. that the torsion is zero, i.e. that the second covariant derivatives of a scalar commute.

In fact, it turns out that these two conditions uniquely determine the $\tilde{\Gamma}$ to be the Christoffel symbols. The second condition implies that the $\tilde{\Gamma}^{\mu}_{\ \nu\lambda}$ are symmetric in the two lower indices,

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]\phi = 0 \quad \Leftrightarrow \quad \tilde{\Gamma}^{\lambda}_{\ \mu\nu} = \tilde{\Gamma}^{\lambda}_{\ \nu\mu} \quad . \tag{4.70}$$

The first condition now allows one to express the $\tilde{\Gamma}^{\lambda}_{\ \mu\nu}$ in terms of the derivatives of the metric, leading uniquely to the familiar expression for the Christoffel symbols $\Gamma^{\mu}_{\ \nu\lambda}$: First of all, by definition / construction one has (e.g. from demanding the Leibniz rule for $\tilde{\nabla}_\mu$)

$$\begin{aligned} \tilde{\nabla}_\mu g_{\nu\lambda} &= \partial_\mu g_{\nu\lambda} - \tilde{\Gamma}^{\rho}_{\ \mu\nu} g_{\rho\lambda} - \tilde{\Gamma}^{\rho}_{\ \mu\lambda} g_{\nu\rho} \\ &\equiv \partial_\mu g_{\nu\lambda} - \tilde{\Gamma}_{\lambda\mu\nu} - \tilde{\Gamma}_{\nu\mu\lambda} \quad . \end{aligned} \tag{4.71}$$

Requiring that this be zero implies in particular that

$$\begin{aligned} 0 &= \tilde{\nabla}_\mu g_{\nu\lambda} + \tilde{\nabla}_\nu g_{\mu\lambda} - \tilde{\nabla}_\lambda g_{\mu\nu} \\ &= \partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} - \tilde{\Gamma}_{\lambda\mu\nu} - \tilde{\Gamma}_{\nu\mu\lambda} - \tilde{\Gamma}_{\lambda\nu\mu} - \tilde{\Gamma}_{\mu\nu\lambda} + \tilde{\Gamma}_{\nu\lambda\mu} + \tilde{\Gamma}_{\mu\lambda\nu} \\ &= 2(\Gamma_{\lambda\mu\nu} - \tilde{\Gamma}_{\lambda\mu\nu}) \end{aligned} \tag{4.72}$$

(where the cancellations are entirely due to the assumed symmetry of the coefficients in the last two indices). Thus $\tilde{\Gamma} = \Gamma$. This unique metric-compatible and torsion-free connection is also known as the *Levi-Civita connection.* It is the connection canonically associated to a space-time (manifold) equipped with a metric tensor.

It is of course possible to relax either of the conditions (1) or (2), or both of them. Relaxing (1), however, is probably physically not very meaningful (cf. the argument in section 4.5 for $\nabla_\mu g_{\nu\lambda} = 0$ based on the existence of freely falling coordinate systems, i.e. the equivalence principle).

It is possible, however, to relax (2), and such connections with torsion are popular in certain circles and play a role in certain generalised theories of gravity. In general, for such a connection, the notions of geodesics and autoparallels no longer coincide. However, this difference disappears if Δ happens to be antisymmetric in its lower indices, as one then has

$$\ddot{x}^\mu + \Gamma^\mu_{\ \nu\lambda}\dot{x}^\nu\dot{x}^\lambda = \ddot{x}^\mu + \tilde{\Gamma}^\mu_{\ \nu\lambda}\dot{x}^\nu\dot{x}^\lambda \ , \tag{4.73}$$

so that the presence of torsion may not readily be experimentally detectable.

5 Physics in a Gravitational Field

5.1 The Principle of Minimal Coupling

The fact that the covariant derivative ∇ maps tensors to tensors and reduces to the ordinary partial derivative in a locally inertial coordinate system suggests the following algorithm for obtaining equations that satisfy the Principle of General Covariance:

1. Write down the Lorentz invariant equations of Special Relativity (e.g. those of relativistic mechanics, Maxwell theory, relativistic hydrodynamics, ...).
2. Wherever the Minkowski metric $\eta_{\mu\nu}$ appears, replace it by $g_{\mu\nu}$.
3. Wherever a partial derivative ∂_μ appears, replace it by the covariant derivative ∇_μ (in particular, for the proper-time derivative along a curve this entails replacing $d/d\tau$ by $D/D\tau$)
4. Wherever an integral $\int d^4x$ appears, replace it by $\int \sqrt{g} d^4x$

By construction, these equations are tensorial (generally covariant) and true in the absence of gravity and hence satisfy the Principle of General Covariance. Hence they will be true in the presence of gravitational fields (at least on scales small compared to that of the gravitational fields - if one considers higher derivatives of the metric tensor then there are other equations that one can write down, involving e.g. the curvature tensor, that are tensorial but reduce to the same equations in the absence of gravity).

5.2 Particle Mechanics in a Gravitational Field Revisited

We can see the power of the formalism we have developed so far by rederiving the laws of particle mechanics in a general gravitational field. In Special Relativity, the motion of a particle with mass m moving under the influence of some external force is governed by the equation

$$\text{SR:} \quad \frac{dX^\mu}{d\tau} = \frac{f^\mu}{m} \ , \tag{5.1}$$

where f^μ is the force four-vector and $X^\mu = \dot{x}^\mu$. Thus, using the principle of minimal coupling, the equation in a general gravitational field is

$$\text{GR:} \quad \frac{DX^\mu}{D\tau} = \frac{f^\mu}{m} \ , \tag{5.2}$$

Of course, the left hand side is just the familiar geodesic equation, but we see that it follows much faster from demanding general covariance (the principle of minimal coupling) than from our previous considerations.

5.3 Klein-Gordon Scalar Field in a Gravitational Field

Here is where the formalism we have developed really pays off. We will see once again that, using the minimal coupling rule, we can immediately rewrite the equations for a scalar field (here) and the Maxwell equations (below) in a form in which they are valid in an arbitrary gravitational field.

The action for a (real) free massive scalar field ϕ in Special Relativity is

$$\text{SR:} \qquad S[\phi] = \int d^4x \left[-\tfrac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \tfrac{1}{2}m^2\phi^2\right] \ . \tag{5.3}$$

To covariantise this, we replace $d^4x \to \sqrt{g}d^4x$, $\eta^{\alpha\beta} \to g^{\alpha\beta}$, and we could replace $\partial_\alpha \to \nabla_\alpha$ (but this makes no difference on scalars). Therefore, the covariant action in a general gravitational field is

$$\text{GR:} \qquad S[\phi, g_{\alpha\beta}] = \int \sqrt{g}d^4x \left[-\tfrac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \tfrac{1}{2}m^2\phi^2\right] \ . \tag{5.4}$$

Here I have also indicated the dependence of the action on the metric $g_{\alpha\beta}$. This is not (yet) a dynamical field, though, just the gravitational background field.

The equations of motion for ϕ one derives from this are

$$\frac{\delta}{\delta\phi}S[\phi, g_{\alpha\beta}] = 0 \quad \Rightarrow \quad \left(\Box_g - m^2\right)\phi = 0 \tag{5.5}$$

where $\Box_g = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$, the Laplacian associated to the metric $g_{\alpha\beta}$. This is precisely what one would have obtained by applying the minimal coupling description to the Minkowski Klein-Gordon equation $(\Box_\eta - m^2)\phi = 0$. [If the relative sign of $\Box_\eta$ and m^2 in this equation looks unfamiliar to you, then this is probably due to the fact that in a course where you first encountered the Klein-Gordon equation the opposite (particle physicists') sign convention for the Minkowski metric was used ...]

A comment on how to derive this: if one thinks of the ∂_α in the action as covariant derivatives, $\partial_\alpha \to \nabla_\alpha$, then the calculation is identical to that in Minkowski space provided that one remembers that $\nabla_\alpha\sqrt{g} = 0$. If one sticks with the ordinary partial derivatives, then upon the usual integration by parts one picks up a term $\sim \partial_\alpha(\sqrt{g}g^{\alpha\beta}\partial_\beta\phi)$ which then evidently leads to the Laplacian in the form (4.49).

The energy-momentum tensor of a Klein-Gordon scalar field in Minkowski space is

$$\text{SR:} \quad T_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi + \eta_{\alpha\beta}L = \partial_\alpha\phi\partial_\beta\phi - \tfrac{1}{2}\eta_{\alpha\beta}\left(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right) \tag{5.6}$$

normalised so as to give for T_{00} the positive energy density

$$T_{00} = \tfrac{1}{2}(\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2) \ . \tag{5.7}$$

This energy-momentum tensor is conserved for ϕ a solution to the equations of motion,

$$\left(\Box_\eta - m^2\right)\phi = 0 \quad \Rightarrow \quad \partial^\alpha T_{\alpha\beta} = 0 \ . \tag{5.8}$$

The corresponding energy-momentum tensor in a gravitational field is then evidently

$$\text{GR:}\quad T_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi + g_{\alpha\beta}L = \partial_\alpha\phi\partial_\beta\phi - \tfrac{1}{2}g_{\alpha\beta}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right) \tag{5.9}$$

and it is covariantly conserved for ϕ a solution to the equations of motion in a gravitational background,

$$\left(\Box_g - m^2\right)\phi = 0 \quad \Rightarrow \quad \nabla^\alpha T_{\alpha\beta} = 0 \ . \tag{5.10}$$

Note that we cannot expect to have an ordinary conservation law because there is no translation invariance in a general gravitational background. We will come back to this issue below and in section 6. In particular, we cannot even derive the energy-momentum tensor (5.9) via Noether's theorem (applied to translations).

We can, however, obtain it in a quite different (and perhaps at this point rather mysterious) way, namely by varying the action $S[\phi, g_{\alpha\beta}]$ not with respect to ϕ but with respect to the metric! Indeed, recalling the formula for the derivative / variation of the determinant of the metric from section 4.6,

$$\delta g = g^{-1}g^{\alpha\beta}\delta g_{\alpha\beta} = -g g_{\alpha\beta}\delta g^{\alpha\beta} \tag{5.11}$$

(the second quality following from $\delta(g_{\alpha\beta}g^{\alpha\beta}) = 0$), one finds that under $g^{\alpha\beta} \to g^{\alpha\beta} + \delta g^{\alpha\beta}$ the action varies by

$$\delta S[\phi, g_{\alpha\beta}] = -\tfrac{1}{2}\int \sqrt{g}d^4x\ T_{\alpha\beta}\delta g^{\alpha\beta} \tag{5.12}$$

or

$$T_{\alpha\beta} = -\frac{2}{\sqrt{g}}\frac{\delta S[\phi, g_{\alpha\beta}]}{\delta g^{\alpha\beta}} \ . \tag{5.13}$$

More should (and will) be said about this in due course ...

5.4 Maxwell Theory in a Gravitational Field

Given the vector potential A_μ, the Maxwell field strength tensor in Special Relativity is

$$\text{SR:}\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{5.14}$$

Therefore in a general metric space time (gravitational field) we have

$$\text{GR:}\quad F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{5.15}$$

Actually, this is a bit misleading. The field strength tensor (two-form) in any, Abelian or non-Abelian, gauge theory is always given in terms of the gauge-covariant exterior derivative of the vector potential (connection), and as such has nothing whatsoever to do with the metric on space-time. So you should not really regard the first equality in the above equation as the definition of $F_{\mu\nu}$, but you should regard the second equality as a proof that $F_{\mu\nu}$, *always* defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, is a tensor. The mistake of adopting $\nabla_\mu A_\nu - \nabla_\nu A_\mu$ as the definition of $F_{\mu\nu}$ in a curved space time has led some poor souls to believe, and even claim in published papers, that in a space time with torsion, for which the second equality does not hold, the Maxwell field strength tensor is modified by the torsion. This is nonsense.

In Special Relativity, the Maxwell equations read

$$\begin{aligned} \text{SR:} \qquad & \partial_\mu F^{\mu\nu} = -J^\nu \\ & \partial_{[\mu} F_{\nu\lambda]} = 0 \ . \end{aligned} \tag{5.16}$$

Thus in a general gravitational field (curved space time) these equations become

$$\begin{aligned} \text{GR:} \qquad & \nabla_\mu F^{\mu\nu} = -J^\nu \\ & \nabla_{[\mu} F_{\nu\lambda]} = 0 \ , \end{aligned} \tag{5.17}$$

where now of course all indices are raised and lowered with the metric $g_{\mu\nu}$,

$$F^{\mu\nu} = g^{\mu\lambda} g^{\nu\rho} F_{\lambda\rho} \ . \tag{5.18}$$

Regarding the use of the covariant derivative in the second equation, the same *caveat* as above applies.

Using the results derived above, we can rewrite these two equations as

$$\begin{aligned} \text{GR:} \qquad & \partial_\mu (g^{1/2} F^{\mu\nu}) = -g^{1/2} J^\nu \\ & \partial_{[\mu} F_{\nu\lambda]} = 0 \ . \end{aligned} \tag{5.19}$$

The electromagnetic force acting on a particle of charge e is given in Special Relativity by

$$\text{SR:} \quad f^\mu = e F^\mu_{\ \nu} \dot{x}^\nu \ . \tag{5.20}$$

Thus in General Relativity it becomes

$$\text{GR:} \quad f^\mu = e g^{\mu\lambda} F_{\lambda\nu} \dot{x}^\nu \ . \tag{5.21}$$

The Lorentz-invariant action of (vacuum) Maxwell theory is

$$\text{SR:} \quad S[A_\alpha] = -\tfrac{1}{4} \int d^4x \ F_{\mu\nu} F^{\mu\nu} \tag{5.22}$$

$$\text{GR:} \quad S[A_\alpha, g_{\alpha\beta}] = -\tfrac{1}{4} \int \sqrt{g} d^4x \ F_{\mu\nu} F^{\mu\nu} \equiv -\tfrac{1}{4} \int \sqrt{g} d^4x \ g^{\mu\lambda} g^{\nu\rho} F_{\mu\nu} F_{\lambda\rho} \tag{5.23}$$

(and to add sources, one adds $\int \sqrt{g} d^4x \ A_\mu J^\mu$). By varying this action with respect to the A_μ one obtains the vacuum Maxwell equations $\nabla_\mu F^{\mu\nu} = 0$,

$$\frac{\delta}{\delta A_\nu} S[A_\alpha, g_{\alpha\beta}] = 0 \quad \Rightarrow \quad \nabla_\mu F^{\mu\nu} = 0 \ . \tag{5.24}$$

Instead of just adding a source-term by hand, a more coherent approach (which also provides the sources with their own dynamics) is to consider a matter action (minimally) coupled to the Maxwell field,

$$S_M[\phi] \to S_M[\phi, A_\alpha] \ . \tag{5.25}$$

The combined Maxwell + matter action will then give rise to the Maxwell equations with a source provided that one defines the current J^α as the variation of the matter action with respect to the gauge field,

$$J^\alpha \sim \frac{\delta S_M[\phi, A_\alpha]}{\delta A_\alpha} \ . \tag{5.26}$$

By the same rationale, we should define the source term for the gravitational field by the variation of the gravitationally minimally coupled matter action with respect to the metric. But the source term for the gravitational field equation should be the energy-momentum tensor. We have already seen that this works out correctly in the case of a scalar field. Let us now see what we get here.

Recall that in the case of Maxwell theory in Minkoswki space-time the canonical Noether energy-momentum tensor (associated to the translation invariance of the Maxwell action)

$$\Theta_{\mu\nu} = -\frac{\partial L}{\partial(\partial^\mu A_\lambda)}\partial_\nu A_\lambda + \eta_{\mu\nu} L = F_\mu^{\ \lambda}\partial_\nu A_\lambda - \tfrac{1}{4}\eta_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma} \tag{5.27}$$

is (of course) conserved but neither symmetric nor gauge-invariant. This can be rectified by adding an identically conserved correction-term and/or by considering Lorentz transformations as well, not just translations, and taking into account the tensorial (non-scalar) nature of the vector potential under Lorentz-transformations. Either way one then finally arrives at the covariant (improved, symmetric and gauge-invariant) energy-momentum tensor of Maxwell theory, namely

$$\text{SR:} \quad T_{\mu\nu} = F_{\mu\lambda}F_\nu^{\ \lambda} + \eta_{\mu\nu}L = F_{\mu\lambda}F_\nu^{\ \lambda} - \tfrac{1}{4}\eta_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma} \ , \tag{5.28}$$

normalised so as to give for T_{00} the positive definite energy density

$$T_{00} = \tfrac{1}{2}(\vec{E}^2 + \vec{B}^2) \ . \tag{5.29}$$

Therefore according to the minimal couplling description, the energy-momentum tensor in a gravitational field is

$$\text{GR:} \quad T_{\mu\nu} = F_{\mu\lambda}F_\nu^{\ \lambda} - \tfrac{1}{4}g_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma} \ , \tag{5.30}$$

where all indices are raised with the inverse metric $g^{\mu\nu}$. The (non-)conservation equation

$$\text{SR:} \quad \partial_\mu T^{\mu\nu} = J_\mu F^{\mu\nu} \ , \tag{5.31}$$

(deriving this requires using both sets of Maxwell equations) becomes the covariant (non-)conservation law

$$\text{GR:} \quad \nabla_\mu T^{\mu\nu} = J_\mu F^{\mu\nu} \quad , \tag{5.32}$$

Once again we observe the remarkable fact that this energy-momentum tensor can be obtained by varying the action with respect to the metric,

$$T_{\alpha\beta} = -\frac{2}{\sqrt{g}} \frac{\delta S[A_\alpha, g_{\alpha\beta}]}{\delta g^{\alpha\beta}} \quad . \tag{5.33}$$

This procedure automatically, and in general, gives rise to a symmetric and gauge invariant tensor by construction, without the need for "correction terms", clearly an appealing feature. We will see later (section 10.3) that it is also automatically covariantly conserved "on-shell" by virtue of the general covariance of the action.

5.5 On the Energy-Momentum Tensor for Weyl-invariant Actions

Another general feature of the energy-momentum tensor that is readily understood by adopting the definition

$$T_{\alpha\beta} = -\frac{2}{\sqrt{g}} \frac{\delta S_{\text{matter}}}{\delta g^{\alpha\beta}} \quad . \tag{5.34}$$

is the relation between Weyl invariance and the trace of the energy-momentum tensor.

We consider the situation where the minimally coupled matter action happens to be invariant under *Weyl rescalings*, i.e. under rescalings of the metric

$$g_{\alpha\beta}(x) \to \mathrm{e}^{2\omega(x)} g_{\alpha\beta}(x) \tag{5.35}$$

by a positive definite function, or infinitesimally

$$\delta_\omega g_{\alpha\beta}(x) = 2\omega(x) g_{\alpha\beta}(x) \quad . \tag{5.36}$$

Examples of such actions are e.g. the action of a massless scalar field (5.4) in $d = 2$ (space-time) dimensions

$$S[\phi, g_{\alpha\beta}] = -\tfrac{1}{2} \int d^2x \, \sqrt{g} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \tag{5.37}$$

and that of Maxwell theory (5.23) in $d = 4$ dimensions,

$$S[A_\alpha, g_{\alpha\beta}] = -\tfrac{1}{4} \int d^4x \, \sqrt{g} g^{\alpha\beta} g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} \quad . \tag{5.38}$$

Indeed, in that case the metric dependence of the action is precisely such that the combination of of the determinant $\sqrt{g}$ and the inverse metric that appears is invariant under Weyl rescalings,

$$g_{\alpha\beta} \to \mathrm{e}^{2\omega} g_{\alpha\beta} \quad \Rightarrow \quad \left\{ \begin{array}{ll} d = 2 & \sqrt{g} g^{\alpha\beta} \to \sqrt{g} g^{\alpha\beta} \\ d = 4 & \sqrt{g} g^{\alpha\beta} g^{\gamma\delta} \to \sqrt{g} g^{\alpha\beta} g^{\gamma\delta} \end{array} \right. \tag{5.39}$$

This is reflected in the fact that the corresponding energy-momentum tensor is traceless precisely in these dimensions: from (5.9) and (5.30) one finds

$$
\begin{aligned}
T_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi - \tfrac{1}{2}g_{\alpha\beta}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) \quad &\Rightarrow \quad T^\alpha_{\ \alpha} = -\tfrac{1}{2}(d-2)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\
T_{\alpha\beta} = F_{\alpha\lambda}F_\beta^{\ \lambda} - \tfrac{1}{4}g_{\alpha\beta}F_{\lambda\sigma}F^{\lambda\sigma} \quad &\Rightarrow \quad T^\alpha_{\ \alpha} = -\tfrac{1}{4}(d-4)F_{\mu\nu}F^{\mu\nu} \ .
\end{aligned}
\tag{5.40}
$$

The relation between these two observations / assertions is provided by noting that if the matter action is invariant under Weyl rescalings one has

$$
\begin{aligned}
0 = \delta_\omega S_{\text{matter}} &= -\tfrac{1}{2}\int \sqrt{g}d^dx\ T_{\alpha\beta}(x)\delta_\omega g^{\alpha\beta}(x) \\
&= \int \sqrt{g}d^dx\ T_{\alpha\beta}(x)g^{\alpha\beta}(x)\omega(x) = \int \sqrt{g}d^dx\ T^\alpha_{\ \alpha}(x)\omega(x) \ .
\end{aligned}
\tag{5.41}
$$

Since this is to be zero for all functions $\omega(x)$, this proves

$$
\text{Invariance under Weyl Rescalings} \quad \Rightarrow \quad T^\alpha_{\ \alpha} = 0 \ . \tag{5.42}
$$

5.6 Conserved Quantities from Covariantly Conserved Currents

In Special Relativity a conserved current J^μ is characterised by the vanishing of its divergence, i.e. by $\partial_\mu J^\mu = 0$. It leads to a conserved charge Q by integrating J^μ over a spacelike hypersurface, say the one described by $t = t_0$. This is usually written as something like

$$
Q = \int_{t=t_0} J^\mu dS_\mu \ , \tag{5.43}
$$

where dS_μ is the induced volume element on the hypersurface. That Q is conserved, i.e. independent of t_0, is a consequence of

$$
Q(t_1) - Q(t_0) = \int_V d^4x\ \partial_\mu J^\mu = 0 \ , \tag{5.44}
$$

where V is the four-volume $\mathbb{R}^3 \times [t_0, t_1]$. This holds provided that J vanishes at spatial infinity.

Now in General Relativity, the conservation law will be replaced by the covariant conservation law $\nabla_\mu J^\mu = 0$, and one may wonder if this also leads to some conserved charges in the ordinary sense. The answer is yes because, recalling the formula for the covariant divergence of a vector,

$$
\nabla_\mu J^\mu = g^{-1/2}\partial_\mu(g^{1/2}J^\mu) \ , \tag{5.45}
$$

we see that

$$
\nabla_\mu J^\mu = 0 \Leftrightarrow \partial_\mu(g^{1/2}J^\mu) = 0 \ , \tag{5.46}
$$

so that $g^{1/2}J^\mu$ is a conserved current in the ordinary sense. We then obtain conserved quantities in the ordinary sense by integrating J^μ over a spacelike hypersurface Σ. Using the generalised Gauss' theorem appropriate for metric space-times, one can see that Q is invariant under deformations of Σ.

In order to write down more precise equations for the charges in this case, we would have to understand how a metric on space-time induces a metric (and hence volume element) on a spacelike hypersurface. This would require developing a certain amount of formalism, useful for certain purposes in Cosmology and for developing a canonical formalism for General Relativity. But as this lies somewhat outside of the things we will do in this course, I will skip this. Suffice it to say here that the first step would be the introduction of a normalised normal vector n^μ to the hypersurface Σ, $n^\mu n_\mu = -1$ and to consider the object $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$. As $h_{\mu\nu}n^\nu = 0$ while $h_{\mu\nu}X^\nu = g_{\mu\nu}X^\nu$ for any vector X^μ normal to n^μ, $h_{\mu\nu}$ induces a metric and volume element on Σ.

The factor $g^{1/2}$ apearing in the current conservation law can be understood physically. To see what it means, split J^μ into its space-time direction u^μ, with $u^\mu u_\mu = -1$, and its magnitude ρ as

$$J^\mu = \rho u^\mu \quad . \tag{5.47}$$

This defines the average four-velocity of the conserved quantity represented by J^μ and its density ρ measured by an observer moving at that average velocity (rest mass density, charge density, number density, ...). Since u^μ is a vector, in order for J^μ to be a vector, ρ has to be a scalar. Therefore this density is defined as per unit proper volume. The factor of $g^{1/2}$ transforms this into density per coordinate volume and this quantity is conserved (in a comoving coordinate system where $J^0 = \rho, J^i = 0$).

We will come back to this in the context of cosmology later on in this course (see section 16) but for now just think of the following picture (Figure 20): take a balloon, draw lots of dots on it at random, representing particles or galaxies. Next choose some coordinate system on the balloon and draw the coordinate grid on it. Now inflate or deflate the balloon. This represents a time dependent metric, roughly of the form $ds^2 = r^2(t)(d\theta^2 + \sin^2\theta d\phi^2)$. You see that the number of dots per coordinate volume element does not change, whereas the number of dots per unit proper volume will.

5.7 Conserved Quantities from Covariantly Conserved Tensors?

In Special Relativity, if $T^{\mu\nu}$ is the energy-momentum tensor of a physical system, it satisfies an equation of the form

$$\partial_\mu T^{\mu\nu} = G^\nu \quad , \tag{5.48}$$

where G^μ represents the density of the external forces acting on the system. In particular, if there are no external forces, the divergence of the energy-momentum tensor is zero. For example, in the case of Maxwell theory and a current corresponding to a charged particle we have

$$G^\nu = J_\mu F^{\mu\nu} = -F^\nu_{\ \mu} J^\mu = -eF^\nu_{\ \mu}\dot{x}^\mu \quad , \tag{5.49}$$

which is indeed the relevant external (Lorentz) force density.

Now, in General Relativity we will instead have

$$\nabla_\mu T^{\mu\nu} = G^\nu \Leftrightarrow g^{-1/2}\partial_\mu(g^{1/2}T^{\mu\nu}) = G^\nu - \Gamma^\nu_{\ \mu\lambda}T^{\mu\lambda} \quad . \tag{5.50}$$

Thus the second term on the right hand side represents the gravitational force density. As expected, it depends on the system on which it acts via the energy momentum tensor. And, as expected, this contribution is not generally covariant.

Now, given a symmetric covariantly conserved energy-momentum tensor, in analogy with Special Relativity, one might like to define quantities like energy and momentum, P^μ, and angular momentum $J^{\mu\nu}$, by integrals of $T^{\mu 0}$ or $x^\mu T^{\nu 0} - x^\nu T^{\mu 0}$ over spacelike hypersurfaces. However, these quantities are rather obviously not covariant, and nor are they conserved. This should perhaps not be too surprising because in Minkowski space these quantities are preserved as a consequence of Poincaré invariance, i.e. because of the symmetries (isometries) of the Minkowski metric.

As a generic metric will have no such isometries, we do not expect to find associated conserved quantities in general. However, if there are symmetries then one can indeed define conserved quantities (think of Noether's theorem), one for each symmetry generator. In order to implement this we need to understand how to define and detect isometries of the metric. For this we need the concepts of Lie derivatives and Killing vectors.

Alternatively, one might try to construct a covariant current-like object by contracting the energy-momentum tensor not with the coordinates but with a vector field V^λ, along the lines of

$$J^\mu_V = T^\mu_{\ \lambda}V^\lambda \quad . \tag{5.51}$$

This is clearly a vector, but is it conserved? Calculating its covariant divergence, and using the fact that $T^{\mu\nu}$ is symmetric and conserved, one finds

$$\nabla_\mu J^\mu_V = \tfrac{1}{2}T^{\mu\nu}(\nabla_\mu V_\nu + \nabla_\nu V_\mu) \quad . \tag{5.52}$$

Thus we would have a conserved current (and associated conserved charge by the previous section) if the vector field V^λ were such that it satisfies $\nabla_\mu V_\nu + \nabla_\nu V_\mu = 0$. The link between this observation and the one in the preceding paragraph regarding symmetries is that, first of all, this is precisely the condition we found in (2.63), as reformulated in (4.58), to generate a symmetry of the metric leading to a conserved charge for geodesics.

More generally, as we will discuss in the next section, vector fields satsifying the equation $\nabla_\mu V_\nu + \nabla_\nu V_\mu = 0$ are indeed in one-to-one correspondence with infinitesimal generators of continuous symmetries of a metric (isometries), giving a pleasing and coherent overall picture of symmetries and conservation laws in a gravitational field.

6 The Lie Derivative, Symmetries and Killing Vectors

6.1 Symmetries of a Metric (Isometries): Preliminary Remarks

Before trying to figure out how to detect symmetries of a metric, or so-called *isometries*, let us decide what we mean by symmetries of a metric. For example, we would say that the Minkowski metric has the Poincaré group as a group of symmetries, because the corresponding coordinate transformations leave the metric invariant.

Likewise, we would say that the standard metrics on the two- or three-sphere have rotational symmetries because they are invariant under rotations of the sphere. We can look at this in one of two ways: either as an *active* transformation, in which we rotate the sphere and note that nothing changes, or as a *passive* transformation, in which we do not move the sphere, all the points remain fixed, and we just rotate the coordinate system. So this is tantamount to a relabelling of the points. From the latter (passive) point of view, the symmetry is again understood as an invariance of the metric under a particular family of coordinate transformations.

Thus consider a metric $g_{\mu\nu}(x)$ in a coordinate system $\{x^\mu\}$ and a change of coordinates $x^\mu \to y^\mu(x^\nu)$ (for the purposes of this and the following section it will be convenient not to label the two coordinate systems by different sets of indices). Of course, under such a coordinate transformation we get a new metric $g'_{\mu\nu}$, with

$$g'_{\mu\nu}(y(x)) = \frac{\partial x^\rho}{\partial y^\mu}\frac{\partial x^\lambda}{\partial y^\nu} g_{\rho\lambda}(x) \quad . \tag{6.1}$$

Since here we do not distinguish coordinate indices associated to different coordinate systems, we now momentarily put primes on the objects themselves in order to keep track of what we are talking about. However, this by itself has nothing to do with possible symmetries of the metric.

Thinking actively, in order to detect symmetries, we should e.g. compare the geometry, given by the line-element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, at two different points x and y related by $y^\mu(x)$. Thus we are led to consider the difference

$$g_{\mu\nu}(y)dy^\mu dy^\nu - g_{\mu\nu}(x)dx^\mu dx^\nu \quad . \tag{6.2}$$

Using the invariance of the line-element under coordinate transformations, i.e. the usual tensorial transformation behaviour of the components of the metric, we see that we can also write this as the difference

$$(g_{\mu\nu}(y) - g'_{\mu\nu}(y))dy^\mu dy^\nu \quad . \tag{6.3}$$

Thus we deduce that what we mean by a symmetry, i.e. invariance of the metric under a coordinate transformation, is the statement

$$g'_{\mu\nu}(y) = g_{\mu\nu}(y) \quad . \tag{6.4}$$

From the passive point of view, in which a coordinate transformation represents a relabelling of the points of the space, this equation compares the new metric at a point P' (with coordinates y^μ) with the old metric at the point P which has the same values of the old coordinates as the point P' has in the new coordinate system, $y^\mu(P') = x^\mu(P)$.

The above equality then states that the new metric at the point P' has the same functional dependence on the new coordinates as the old metric on the old coordinates at the point P. Thus a neighbourhood of P' in the new coordinates looks identical to a neighbourhood of P in the old coordinates, and they can be mapped into each other *isometrically*, i.e. such that all the metric properties, like distances, are preserved. Thus either actively or passively one is led to the above condition.

Note that to detect a continuous symmetry in this way, we only need to consider infinitesimal coordinate transformations. In that case, the above amounts to the statement that metrically the space time looks the same when one moves infinitesimally in the direction given by the coordinate transformation.

6.2 The Lie Derivative for Scalars

We now want to translate the above discussion into a condition for an infinitesimal coordinate transformation

$$x^\mu \to y^\mu(x) = x^\mu + \epsilon V^\mu(x) \tag{6.5}$$

to generate a symmetry of the metric. Here you can and should think of V^μ as a vector field because, even though coordinates themselves of course do not transform like vectors, their infinitesimal variations δx^μ do,

$$z^{\mu'} = z^{\mu'}(x) \to \delta z^{\mu'} = \frac{\partial z^{\mu'}}{\partial x^\mu}\delta x^\mu \tag{6.6}$$

and we think of δx^μ as ϵV^μ.

In fact, we will do something slightly more general than just trying to detect symmetries of the metric. After all, we can also speak of functions or vector fields with symmetries, and this can be extended to arbitrary tensor fields (although that may be harder to visualize). So, for a general tensor field T we will want to compare $T'(y(x))$ with $T(y(x))$ - this is of course equivalent to, and only technically a bit easier than, comparing $T'(x)$ with $T(x)$.

As usual, we start the discussion with scalars. In that case, we want to compare $\phi(y(x))$ with $\phi'(y(x)) = \phi(x)$. We find

$$\phi(y(x)) - \phi'(y(x)) = \phi(x + \epsilon V) - \phi(x) = \epsilon V^\mu \partial_\mu \phi + \mathcal{O}(\epsilon^2) \ . \tag{6.7}$$

We now define the *Lie derivative* of ϕ along the vector field V^μ to be

$$L_V \phi := \lim_{\epsilon \to 0} \frac{\phi(y(x)) - \phi'(y(x))}{\epsilon} \ . \tag{6.8}$$

Evaluating this, we find

$$L_V\phi = V^\mu\partial_\mu\phi \ . \tag{6.9}$$

Thus for a scalar, the Lie derivative is just the ordinary directional derivative, and this is as it should be since saying that a function has a certain symmetry amounts to the assertion that its derivative in a particular direction vanishes.

6.3 The Lie Derivative for Vector Fields

We now follow the same procedure for a vector field W^μ. We will need the matrix $(\partial y^\mu/\partial x^\nu)$ and its inverse for the above infinitesimal coordinate transformation. We have

$$\frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu_{\ \nu} + \epsilon\partial_\nu V^\mu \ , \tag{6.10}$$

and

$$\frac{\partial x^\mu}{\partial y^\nu} = \delta^\mu_{\ \nu} - \epsilon\partial_\nu V^\mu + \mathcal{O}(\epsilon^2) \ . \tag{6.11}$$

Thus we have

$$\begin{aligned} W'^\mu(y(x)) &= \frac{\partial y^\mu}{\partial x^\nu} W^\nu(x) \\ &= W^\mu(x) + \epsilon W^\nu(x)\partial_\nu V^\mu(x) \ , \end{aligned} \tag{6.12}$$

and

$$W^\mu(y(x)) = W^\mu(x) + \epsilon V^\nu\partial_\nu W^\mu(x) + \mathcal{O}(\epsilon^2) \ . \tag{6.13}$$

Hence, defining the Lie derivative $L_V W$ of W by V by

$$L_V W^\mu := \lim_{\epsilon\to 0} \frac{W^\mu(y(x)) - W'^\mu(y(x))}{\epsilon} \ , \tag{6.14}$$

we find

$$L_V W^\mu = V^\nu\partial_\nu W^\mu - W^\nu\partial_\nu V^\mu \ . \tag{6.15}$$

There are several important things to note about this expression:

1. The result looks non-covariant, i.e. non-tensorial. But as a difference of two vectors at the same point (recall the limit $\epsilon \to 0$) the result should again be a vector. This is indeed the case. One way to make this manifest is to rewrite (6.15) in terms of covariant derivatives, as

$$\begin{aligned} L_V W^\mu &= V^\nu\nabla_\nu W^\mu - W^\nu\nabla_\nu V^\mu \\ &= \nabla_V W^\mu - \nabla_W V^\mu \ . \end{aligned} \tag{6.16}$$

   This shows that $L_W W^\mu$ is again a vector field. Note, however, that the Lie derivative, in contrast to the covariant derivative, is defined without reference to any metric.

2. There is an alternative, and perhaps more intuitive, derivation of the above expression (6.15) for the Lie derivative of a vector field along a vector field, which makes both its tensorial character and its interpretation manifest (and which also generalises to other tensor fields; in fact we had already applied it to the metric in section 2.5 to deduce (2.62)).

   Namely, let us assume that we are initially in a coordinate system $\{y^{\mu'}\}$ adapted to V in the sense that $V = \partial/\partial y^a$ for some particular a, i.e. $V^{\mu'} = \delta^{\mu'}_a$ (so that we are locally choosing the flow-lines of V as one of the coordinate lines). In this coordinate system we would naturally define the change of a vector field $W^{\mu'}$ along V as the partial derivative of W along y^a,

$$L_V W^{\mu'} := \frac{\partial}{\partial y^a} W^{\mu'} \quad . \tag{6.17}$$

   We now consider an arbitrary coordinate transformation $x^\alpha = x^\alpha(y^{\mu'})$, and require that $L_V W$ transforms as a vector under coordinate transformations. This will then give us the expression for $L_V W$ in an arbitrary coordinate system:

$$\begin{aligned} \frac{\partial}{\partial y^a} W^{\mu'} &= \frac{\partial x^\alpha}{\partial y^a}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial y^{\mu'}}{\partial x^\beta} W^\beta\right) \\ &\stackrel{!}{=} \frac{\partial y^{\mu'}}{\partial x^\alpha}(L_V W)^\alpha \quad . \end{aligned} \tag{6.18}$$

   Disentangling this, using $V^\alpha = \partial x^\alpha / \partial y^a$ and

$$\frac{\partial x^\alpha}{\partial y^a}\frac{\partial^2 y^{\mu'}}{\partial x^\alpha \partial x^\beta} = \frac{\partial x^\alpha}{\partial y^a}\frac{\partial}{\partial x^\beta}\frac{\partial y^{\mu'}}{\partial x^\alpha} = -\frac{\partial V^\alpha}{\partial x^\beta}\frac{\partial y^{\mu'}}{\partial x^\alpha} \quad , \tag{6.19}$$

   one recovers the definition (6.15).

3. Note that (6.15) is antisymmetric in V and W. Hence it defines a commutator $[V, W]$ on the space of vector fields,

$$[V, W]^\mu := L_V W^\mu = -L_W V^\mu \quad . \tag{6.20}$$

   This is actually a Lie bracket, i.e. it satisfies the Jacobi identity

$$[V, [W, X]]^\mu + [X, [V, W]]^\mu + [W, [X, V]]^\mu = 0 \quad . \tag{6.21}$$

   This can also be rephrased as the statement that the Lie derivative is also a derivation of the Lie bracket, i.e. that one has

$$L_V [W, X]^\mu = [L_V W, X]^\mu + [W, L_V X]^\mu \quad . \tag{6.22}$$

4. I want to reiterate at this point that it is extremely useful to think of vector fields as first order linear differential operators, via $V^\mu \to V = V^\mu \partial_\mu$. In this case, the

Lie bracket $[V,W]$ is simply the ordinary commutator of differential operators,

$$\begin{aligned}
[V,W] &= [V^\mu\partial_\mu, W^\nu\partial_\nu] \\
&= V^\mu(\partial_\mu W^\nu)\partial_\nu + V^\mu W^\nu\partial_\mu\partial_\nu - W^\nu(\partial_\nu V^\mu)\partial_\mu - W^\nu V^\mu\partial_\nu\partial_\nu \\
&= (V^\nu\partial_\nu W^\mu - W^\nu\partial_\nu V^\mu)\partial_\mu \\
&= (L_V W)^\mu\partial_\mu = [V,W]^\mu\partial_\mu \ .
\end{aligned} \tag{6.23}$$

From this point of view, the Jacobi identity is obvious.

5. Having equipped the space of vector fields with a Lie algebra structure, in fact with the structure of an *infinite-dimensional Lie algebra*, it is fair to ask 'the Lie algebra of what group?'. Well, we have seen above that we can think of vector fields as infinitesimal generators of coordinate transformations. Hence, formally at least, the Lie algebra of vector fields is the Lie algebra of the group of coordinate transformations (passive point of view) or diffeomorphisms (active point of view).

6.4 The Lie Derivative for other Tensor Fields

To extend the definition of the Lie derivative to other tensors, we can proceed in one of two ways. We can either extend the above procedure to other tensor fields by defining

$$L_V T^{\cdots}_{\cdots} := \lim_{\epsilon\to 0}\frac{T^{\cdots}_{\cdots}(y(x)) - T'^{\cdots}_{\cdots}(y(x))}{\epsilon} \ . \tag{6.24}$$

Or we can extend it to other tensors by proceeding as in the case of the covariant derivative, i.e. by demanding the Leibniz rule. In either case, the result can be rewritten in manifestly tensorial form in terms of covariant derivatives.

The result is that the Lie derivative of a (p,q)-tensor T is, like the covariant derivative, the sum of three kinds of terms: the directional covariant derivative of T along V, p terms with a minus sign, involving the covariant derivative of V contracted with each of the upper indices, and q terms with a plus sign, involving the convariant derivative of V contracted with each of the lower indices (note that the plus and minus signs are interchanged with respect to the covariant derivative). Thus, e.g., the Lie derivative of a (1,2)-tensor is

$$L_V T^\mu_{\nu\lambda} = V^\rho\nabla_\rho T^\mu_{\nu\lambda} - T^\rho_{\nu\lambda}\nabla_\rho V^\mu + T^\mu_{\rho\lambda}\nabla_\nu V^\rho + T^\mu_{\nu\rho}\nabla_\lambda V^\rho \ . \tag{6.25}$$

The fact that the Lie derivative provides a representation of the Lie algebra of vector fields by first-order differential operators on the space of (p,q)-tensors is expressed by the identity

$$[L_V, L_W] = L_{[V,W]} \ . \tag{6.26}$$

6.5 The Lie Derivative of the Metric and Killing Vectors

The above general formula becomes particularly simple for the metric tensor $g_{\mu\nu}$. The first term is not there (because the metric is covariantly constant), so the Lie derivative is the sum of two terms (with plus signs) involving the covariant derivative of V,

$$L_V g_{\mu\nu} = g_{\lambda\nu}\nabla_\mu V^\lambda + g_{\mu\lambda}\nabla_\nu V^\lambda \ . \tag{6.27}$$

Lowering the index of V with the metric, this can be written more succinctly as

$$L_V g_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu \ . \tag{6.28}$$

We are now ready to return to our discussion of isometries (symmetries of the metric). Evidently, an infinitesimal coordinate transformation is a symmetry of the metric if $L_V g_{\mu\nu} = 0$,

$$V \text{ generates an isometry } \Leftrightarrow \nabla_\mu V_\nu + \nabla_\nu V_\mu = 0 \ . \tag{6.29}$$

Vector fields V satisfying this equation are called *Killing vectors* - not because they kill the metric but after the 19th century mathematician W. Killing.

An alternative way of writing the Killing equation, which is not manifestly covariant but which makes it manifest that only derivatives of the metric in the V-direction (and thus only the corresponding Christoffel symbols) enter, is

$$\nabla_\mu V_\nu + \nabla_\nu V_\mu = 0 \quad \Leftrightarrow \quad V^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu V^\lambda g_{\lambda\nu} + \partial_\nu V^\lambda g_{\mu\lambda} = 0 \ . \tag{6.30}$$

This is precisely the condition (2.63) we had encountered first in our discussion of first integrals of motion for the geodesic equation, and which we had already rewritten in terms of covariant derivatives, as in (6.28) above, in (4.58).

Remarks:

1. Since they are associated with symmetries of space time, and since symmetries are always of fundamental importance in physics, Killing vectors will play an important role in the following. Our most immediate concern will be with the conserved quantities associated with Killing vectors. We will return to a more detailed discussion of Killing vectors and symmetric space times in the context of Cosmology later on. For now, let us just note that by virtue of (6.26) Killing vectors form a Lie algebra, i.e. if V and W are Killing vectors, then also $[V, W]$ is a Killing vector,

$$L_V g_{\mu\nu} = L_W g_{\mu\nu} = 0 \Rightarrow L_{[V,W]} g_{\mu\nu} = 0 \ . \tag{6.31}$$

Indeed one has

$$L_{[V,W]} g_{\mu\nu} = L_V L_W g_{\mu\nu} - L_W L_V g_{\mu\nu} = 0 \ . \tag{6.32}$$

This algebra is (a subalgebra of) the Lie algebra of the isometry group.

2. For example, the collection of all Killing vectors of the Minkowski metric generates the Lie algebra of the Poincaré group. Indeed, for the Minkoski space-time in inertial (Cartesian) coordinates, i.e. with the constant standard metric $\eta_{\alpha\beta}$, the Killing condition simply becomes

$$\partial_\alpha V_\beta + \partial_\beta V_\alpha = 0 \quad , \tag{6.33}$$

which is solved by

$$V^\alpha = \omega^\alpha_\beta \, x^\beta + \epsilon^\alpha \tag{6.34}$$

where the ϵ^α are constant parameters and the constant matrices ω^α_β satisfy $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$. These are precisely the infinitesimal Lorentz transformations and translations of the Poincaré algebra.

3. Another simple example is provided by the two-sphere: as mentioned before, in some obvious sense the standard metric on the two-sphere is rotationally invariant. In particular, with our new terminology we would expect the vector field ∂_ϕ, i.e. the vector field with components $V^\phi = 1, V^\theta = 0$ to be Killing. Let us check this. With the metric $d\theta^2 + \sin^2\theta d\phi^2$, the corresponding covector V_μ, obtained by lowering the indices of the vector field V^μ, are

$$V_\theta = 0 \quad , \qquad V_\phi = \sin^2\theta \quad . \tag{6.35}$$

The Killing condition breaks up into three equations, and we verify

$$\begin{aligned} \nabla_\theta V_\theta &= \partial_\theta V_\theta - \Gamma^\mu_{\theta\theta} V_\mu \\ &= -\Gamma^\phi_{\theta\theta} \sin^2\theta = 0 \\ \nabla_\theta V_\phi + \nabla_\phi V_\theta &= \partial_\theta V_\phi - \Gamma^\mu_{\theta\phi} V_\mu + \partial_\phi V_\theta - \Gamma^\mu_{\theta\phi} V_\mu \\ &= 2\sin\theta\cos\theta - 2\cot\theta\sin^2\theta = 0 \\ \nabla_\phi V_\phi &= \partial_\phi V_\phi + \Gamma^\mu_{\phi\phi} V_\mu = 0 \quad . \end{aligned} \tag{6.36}$$

Alternatively, using the non-covariant form (6.30) of the Killing equation, one finds, since $V^\phi = 1, V^\theta = 0$ are constant, that the Killing equation reduces to

$$\partial_\phi g_{\mu\nu} = 0 \quad , \tag{6.37}$$

which is obviously satisfied. This is clearly a simpler and more efficient argument.

4. In general, if the components of the metric are all independent of a particular coordinate, say y, then by the above argument $V = \partial_y$ is a Killing vector,

$$\partial_y g_{\mu\nu} = 0 \;\; \forall \;\; \mu,\nu \quad \Rightarrow \quad V = \partial_y \;\; \text{is a Killing Vector} \tag{6.38}$$

Such a coordinate system, in which one of the coordinate lines agrees with the integral curves of the Killing vector, is said to be adapted to the Killing vector (or isometry) in question. As we did in section 2.5, one can also take the above equations as the starting point for what one means by a symmetry of the metric (isometry) and then simply transform it to an arbitrary coordinate system by requiring that it transforms as a $(0,2)$-tensor. Then one arrives at the Killing condition in the form (6.30).

6.6 Killing Vectors and Conserved Quantities

We are used to the fact that symmetries lead to conserved quantities (Noether's theorem). For example, in classical mechanics, the angular momentum of a particle moving in a rotationally symmetric gravitational field is conserved. In the present context, the concept of 'symmetries of a gravitational field' is replaced by 'symmetries of the metric', and we therefore expect conserved charges associated with the presence of Killing vectors. Here are the two most important classes of examples of this phenomenon:

1. Killing Vectors, Geodesics and Conserved Quantities

   Let K^μ be a Killing vector field, and $x^\mu(\tau)$ be a geodesic. Then the quantity $K_\mu \dot{x}^\mu$ is constant along the geodesic. Indeed,

$$\begin{aligned} \frac{d}{d\tau}(K_\mu \dot{x}^\mu) &= (\frac{D}{D\tau}K_\mu)\dot{x}^\mu + K_\mu \frac{D}{D\tau}\dot{x}^\mu \\ &= \nabla_\nu K_\mu \dot{x}^\nu \dot{x}^\mu + 0 \\ &= \tfrac{1}{2}(\nabla_\nu K_\mu + \nabla_\mu K_\nu)\dot{x}^\mu \dot{x}^\nu = 0 \ . \end{aligned} \tag{6.39}$$

   Note that this is precisely the conserved quantity Q_V (2.64) with $V \to K$ deduced from Noether's theorem and the variational principle for geodesics in section 2.5

2. Conserved Currents from the Energy-Momentum Tensor

   Let K^μ be a Killing vector field, and $T^{\mu\nu}$ the covariantly conserved symmetric energy-momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$. Then $J^\mu = T^{\mu\nu}K_\nu$ is a covariantly conserved current. Indeed,

$$\begin{aligned} \nabla_\mu J^\mu &= (\nabla_\mu T^{\mu\nu})K_\nu + T^{\mu\nu}\nabla_\mu K_\nu \\ &= 0 + \tfrac{1}{2}T^{\mu\nu}(\nabla_\mu K_\nu + \nabla_\nu K_\mu) = 0 \ . \end{aligned} \tag{6.40}$$

   Hence, as we now have a conserved current, we can associate with it a conserved charge in the way discussed above.

Another situation of interest occurs when one has a theory invariant under Weyl rescalings and thus a traceless energy-momentum tensor (section 5.5). In that case one can associate conserved currents not only to Killing vectors fields but also to *conformal Killing vectors* C^μ, satisfying

$$\nabla_\mu C_\nu + \nabla_\nu C_\mu = 2\omega(x) g_{\mu\nu} \tag{6.41}$$

for some function $\omega(x)$. Such conformal Killing vectors generate coordinate transformations that leave the metric invariant up to an overall (Weyl) rescaling. If the theory is invariant under such Weyl rescalings, then the energy-momntum tensor is traceless and there should also be a corresponding conserved current. Indeed, we have

2' Let C^μ be a conformal Killing vector field, and $T^{\mu\nu}$ a covariantly conserved symmetric and traceless energy-momentum tensor, $\nabla_\mu T^{\mu\nu} = T^{\mu\nu} g_{\mu\nu} = 0$. Then $J^\mu = T^{\mu\nu} C_\nu$ is a covariantly conserved current. Indeed,

$$\begin{aligned} \nabla_\mu J^\mu &= (\nabla_\mu T^{\mu\nu}) C_\nu + T^{\mu\nu} \nabla_\mu C_\nu \\ &= 0 + \tfrac{1}{2} T^{\mu\nu} (\nabla_\mu C_\nu + \nabla_\nu C_\mu) = \omega(x) T^{\mu\nu} g_{\mu\nu} = 0 \ . \end{aligned} \tag{6.42}$$

7 Curvature I: The Riemann Curvature Tensor

7.1 Curvature: Preliminary Remarks

We now come to one of the most important concepts of General Relativity and Riemannian Geometry, that of curvature and how to describe it in tensorial terms. Among other things, this will finally allow us to decide unambiguously if a given metric is just the (flat) Minkowski metric in disguise or the metric of a genuinely curved space. It will also lead us fairly directly to the Einstein equations, i.e. to the field equations for the gravitational field.

Recall that the equations that describe the behaviour of particles and fields in a gravitational field involve the metric and the Christoffel symbols determined by the metric. Thus the equations for the gravitational field should be generally covariant (tensorial) differential equations for the metric.

But at first, here we seem to face a dilemma. How can we write down covariant differential equations for the metric when the covariant derivative of the metric is identically zero? Having come to this point, Einstein himself reached an impasse and required the help of his mathematician friend Marcel Grossmann whom he had asked to investigate if there were any tensors that could be built from the second derivatives of the metric.

Grossmann soon found that this problem had indeed been addressed and solved in the mathematics literature, in particular by Riemann (generalising work of Gauss on curved surfaces), Ricci and Levi-Civita. It was shown by them that there are indeed non-trivial tensors that can be constructed from (ordinary) derivatives of the metric. These can then be used to write down covariant differential equations for the metric.

The most important among these are the Riemann curvature tensor and its various contractions. In fact, it is known that these are the only tensors that can be constructed from the metric and its first and second derivatives, and they will therefore play a central role in all that follows.

7.2 The Riemann Curvature Tensor from the Commutator of Covariant Derivatives

Technically the most straightforward way of introducing the Riemann curvature tensor is via the commutator of covariant derivatives. As this is not geometrically the most intuitive way of introducing the concept of curvature, we will then, once we have defined it and studied its most important algebraic properties, study to which extent the curvature tensor reflects the geometric properties of space time.

As mentioned before, second covariant derivatives do not commute on (p, q)-tensors unless $p = q = 0$. However, the fact that they *do* commute on scalars has the pleasant consequence that e.g. the commutator of covariant derivatives acting on a vector field V^μ does not involve any derivatives of V^μ. In fact, I will first show, without actually calculating the commutator, that

$$[\nabla_\mu, \nabla_\nu]\phi V^\lambda = \phi[\nabla_\mu, \nabla_\nu]V^\lambda \tag{7.1}$$

for any scalar field ϕ. This implies that $[\nabla_\mu, \nabla_\nu]V^\lambda$ cannot depend on derivatives of V because if it did it would also have to depend on derivatives of ϕ. Hence, the commutator can be expressed purely algebraically in terms of V. As the dependence on V is clearly linear, there must therefore be an object $R^\lambda_{\ \sigma\mu\nu}$ such that

$$[\nabla_\mu, \nabla_\nu]V^\lambda = R^\lambda_{\ \sigma\mu\nu}V^\sigma \quad . \tag{7.2}$$

This can of course also be verified by a direct calculation, and we will come back to this below. For now let us just note that, since the left hand side of this equation is clearly a tensor for any V, the quotient theorem implies that $R^\lambda_{\ \sigma\mu\nu}$ has to be a tensor. It is the famous *Riemann-Christoffel Curvature Tensor*.

Let us first verify (7.1). We have

$$\nabla_\mu\nabla_\nu\phi V^\lambda = (\nabla_\mu\nabla_\nu\phi)V^\lambda + (\nabla_\nu\phi)(\nabla_\mu V^\lambda) + (\nabla_\mu\phi)(\nabla_\nu V^\lambda) + \phi\nabla_\mu\nabla_\nu V^\lambda \quad . \tag{7.3}$$

Thus, upon taking the commutator the second and third terms drop out and we are left with

$$\begin{aligned} [\nabla_\mu, \nabla_\nu]\phi V^\lambda &= ([\nabla_\mu, \nabla_\nu]\phi)V^\lambda + \phi[\nabla_\mu, \nabla_\nu]V^\lambda \\ &= \phi[\nabla_\mu, \nabla_\nu]V^\lambda \quad , \end{aligned} \tag{7.4}$$

which is what we wanted to establish.

By explicitly calculating the commutator, one can confirm the structure displayed in (7.2). This explicit calculation shows that the Riemann tensor (for short) is given by

$$\boxed{R^\lambda_{\ \sigma\mu\nu} = \partial_\mu\Gamma^\lambda_{\ \sigma\nu} - \partial_\nu\Gamma^\lambda_{\ \sigma\mu} + \Gamma^\lambda_{\ \mu\rho}\Gamma^\rho_{\ \nu\sigma} - \Gamma^\lambda_{\ \nu\rho}\Gamma^\rho_{\ \mu\sigma}} \tag{7.5}$$

Note how useful the quotient theorem is in this case. It would be quite unpleasant to have to verify the tensorial nature of this expression by explicitly checking its behaviour under coordinate transformations.

Note also that this tensor is clearly zero for the Minkowski metric written in Cartesian coordinates. Hence it is also zero for the Minkowski metric written in any other coordinate system. We will prove the converse, that vanishing of the Riemann curvature tensor implies that the metric is equivalent to the Minkowski metric, below.

It is straightforward to extend the above to an action of the commutator $[\nabla_\mu, \nabla_\nu]$ on arbitrary tensors. For covectors we have, since we can raise and lower the indices with the metric with impunity,

$$
\begin{aligned}
[\nabla_\mu, \nabla_\nu]V_\rho &= g_{\rho\lambda}[\nabla_\mu, \nabla_\nu]V^\lambda \\
&= g_{\rho\lambda}R^\lambda_{\ \sigma\mu\nu}V^\sigma \\
&= R_{\rho\sigma\mu\nu}V^\sigma \\
&= R_\rho^{\ \sigma}{}_{\mu\nu}V_\sigma \quad .
\end{aligned} \tag{7.6}
$$

We will see later that the Riemann tensor is antisymmetric in its first two indices. Hence we can also write

$$
[\nabla_\mu, \nabla_\nu]V_\rho = -R^\sigma_{\ \rho\mu\nu}V_\sigma \quad . \tag{7.7}
$$

The extension to arbitrary (p, q)-tensors now follows the usual pattern, with one Riemann curvature tensor, contracted as for vectors, appearing for each of the p upper indices, and one Riemann curvature tensor, contracted as for covectors, for each of the q lower indices. Thus, e.g. for a (1,1)-tensor $A^\lambda_{\ \rho}$ one would find

$$
[\nabla_\mu, \nabla_\nu]A^\lambda_{\ \rho} = R^\lambda_{\ \sigma\mu\nu}A^\sigma_{\ \rho} - R^\sigma_{\ \rho\mu\nu}A^\lambda_{\ \sigma} \quad . \tag{7.8}
$$

I will give two other versions of the fundamental formula (7.2) which are occasionally useful and used.

1. Instead of looking at the commutator $[\nabla_\mu, \nabla_\nu]$ of two derivatives in the coordinate directions x^μ and x^ν, we can look at the commutator $[\nabla_X, \nabla_Y]$ of two directional covariant derivatives. Evidently, in calculating this commutator one will pick up new terms involving $\nabla_X Y^\mu - \nabla_Y X^\mu$. Comparing with (6.16), we see that this is just $[X, Y]^\mu$. The correct formula for the curvature tensor in this case is

   $$
   ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})V^\lambda = R^\lambda_{\ \sigma\mu\nu}X^\mu Y^\nu V^\sigma \quad . \tag{7.9}
   $$

   Note that, in this sense, the curvature measures the failure of the covariant derivative to provide a representation of the Lie algebra of vector fields.

2. Secondly, one can consider a net of curves $x^\mu(\sigma, \tau)$ parmetrizing, say, a two-dimensional surface, and look at the commutators of the covariant derivatives along the σ- and τ-curves. The formula one obtains in this case (it can be obtained from (7.9) by noting that X and Y commute in this case) is

   $$
   \left(\frac{D^2}{D\sigma D\tau} - \frac{D^2}{D\tau D\sigma}\right)V^\lambda = R^\lambda_{\ \sigma\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}V^\sigma \quad . \tag{7.10}
   $$

A priori, the Riemann tensor has $256 = 4^4$ components in 4 dimensions. However, because of a large number of symmetries, the actual number of independent components is much smaller.

In general, to read off all the symmetries from the formula (7.5) is difficult. One way to simplify things is to look at the Riemann curvature tensor at the origin x_0 of a Riemann normal coordinate system (or some other inertial coordinate system). In that case, all the first derivatives of the metric disappear and only the first two terms of (7.5) contribute. One finds

$$\begin{aligned} R_{\alpha\beta\gamma\delta}(x_0) &= g_{\alpha\lambda}(\partial_\gamma \Gamma^\lambda_{\beta\delta} - \partial_\delta \Gamma^\lambda_{\beta\gamma})(x_0) \\ &= (\partial_\gamma \Gamma_{\alpha\beta\delta} - \partial_\delta \Gamma_{\alpha\beta\gamma})(x_0) \\ &= \tfrac{1}{2}(g_{\alpha\delta},_{\beta\gamma} + g_{\beta\gamma},_{\alpha\delta} - g_{\alpha\gamma},_{\beta\delta} - g_{\beta\delta},_{\alpha\gamma})(x_0) \ . \end{aligned} \tag{7.11}$$

In principle, this expression is sufficiently simple to allow one to read off all the symmetries of the Riemann tensor. However, it is more insightful to derive these symmetries in a different way, one which will also make clear *why* the Riemann tensor has these symmetries.

1. $R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}$

   This is obviously true from the definition or by construction.

2. $R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$

   This is a consequence of the fact that the metric is covariantly constant. In fact, we can calculate

$$\begin{aligned} 0 &= [\nabla_\gamma, \nabla_\delta] g_{\alpha\beta} \\ &= R_\alpha{}^\lambda{}_{\gamma\delta} g_{\lambda\beta} + R_\beta{}^\lambda{}_{\gamma\delta} g_{\alpha\lambda} \\ &= (R_{\alpha\beta\gamma\delta} + R_{\beta\alpha\gamma\delta}) \ . \end{aligned} \tag{7.12}$$

3. $R_{\alpha[\beta\gamma\delta]} = 0 \Leftrightarrow R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0$

   This Bianchi identity is a consequence of the fact that there is no torsion. In fact, applying $[\nabla_\gamma, \nabla_\delta]$ to the covector $\nabla_\beta \phi$, ϕ a scalar, one has

$$\nabla_{[\gamma}\nabla_\delta\nabla_{\beta]}\phi = 0 \Rightarrow R^\lambda_{[\beta\gamma\delta]}\nabla_\lambda\phi = 0 \ . \tag{7.13}$$

   As this has to be true for all scalars ϕ, this implies $R_{\alpha[\beta\gamma\delta]} = 0$ (to see this you could e.g. choose the (locally defined) coordinate functions $\phi(x) = x^\mu$ with $\nabla_\lambda \phi = \delta^\mu_\lambda$).

4. $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$

   This identity, stating that the Riemann tensor is symmetric in its two pairs of indices, is not an independent symmetry but can be deduced from the three other symmetries by some not particularly interesting algebraic manipulations.

We can now count how many independent components the Riemann tensor really has. (1) implies that the second pair of indices can only take $N = (4 \times 3)/2 = 6$ independent values. (2) implies the same for the first pair of indices. (4) thus says that the Riemann curvature tensor behaves like a symmetric (6×6) matrix and therefore has $(6 \times 7)/2 = 21$ components. We now come to the remaining condition (3): if two of the indices in (3) are equal, (3) is equivalent to (4) and (4) we have already taken into account. With all indices unequal, (3) then provides one and only one more additional constraint. We conclude that the total number of independent components is 20.

<u>Remarks:</u>

1. Note that this agrees precisely with our previous counting of how many of the second derivatives of the metric cannot be set to zero by a coordinate transformation: the second derivative of the metric has 100 independent components, to be compared with the $4 \times (4 \times 5 \times 6)/(2 \times 3) = 80$ components of the third derivatives of the coordinates. This also leaves 20 components. We thus see very explicitly that the Riemann curvature tensor contains all the coordinate independent information about the geometry up to second derivatives of the metric. In fact, it can be shown that in a Riemann normal coordinate system one has

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + 0 + \tfrac{1}{3} R_{\mu\lambda\sigma\nu}(x_0)(x - x_0)^\lambda (x - x_0)^\sigma + \mathcal{O}((x - x_0)^3) \ . \tag{7.14}$$

2. Just for the record, I note here that in general dimension n the Riemann tensor has $n^2(n^2 - 1)/12$ independent components. This number arises as

$$\begin{aligned} \frac{n^2(n^2-1)}{12} &= \frac{N(N+1)}{2} - \binom{n}{4} \\ N &= \frac{n(n-1)}{2} \end{aligned} \tag{7.15}$$

   and describes (as above) the number of independent components of a symmetric $(N \times N)$-matrix, now subject to $\binom{n}{4}$ conditions which arise from all the possibilities of choosing 4 out of n possible distinct values for the indices in (3). Just as for $n = 4$, this number of components of the Riemann tensor coincides with the number of second derivatives of the metric minus the number of independent components of the third derivatives of the coordinates,

$$\frac{n(n+1)}{2} \times \frac{n(n+1)}{2} - n \times \frac{n(n+1)(n+2)}{2 \times 3} = \frac{n^2(n^2-1)}{12} \ . \tag{7.16}$$

For $n = 2$ this formula predicts one independent component, and this is as it should be. Rather obviously the only independent non-vanishing component of the Riemann tensor in this case is R_{1212}.

Finally, a word of warning: there are a large number of sign conventions involved in the definition of the Riemann tensor (and its contractions we will discuss below), so whenever reading a book or article, in particular when you want to use results or equations presented there, make sure what conventions are being used and either adopt those or translate the results into some other convention. As a check: the conventions used here are such that $R_{\phi\theta\phi\theta}$ as well as the curvature scalar (to be introduced below) are *positive* for the standard metric on the two-sphere.

7.4 The Ricci Tensor and the Ricci Scalar

The Riemann tensor, as we have seen, is a four-index tensor. For many purposes this is not the most useful object. But we can create new tensors by contractions of the Riemann tensor. Due to the symmetries of the Riemann tensor, there is essentially only one possibility, namely the *Ricci tensor*

$$R_{\mu\nu} := R^{\lambda}{}_{\mu\lambda\nu} = g^{\lambda\sigma} R_{\sigma\mu\lambda\nu} \quad . \tag{7.17}$$

It follows from the symmetries of the Riemann tensor that $R_{\mu\nu}$ is symmetric. Indeed

$$R_{\nu\mu} = g^{\lambda\sigma} R_{\sigma\nu\lambda\mu} = g^{\lambda\sigma} R_{\lambda\mu\sigma\nu} = R^{\sigma}{}_{\mu\sigma\nu} = R_{\mu\nu} \quad . \tag{7.18}$$

Thus, for $n = 4$, the Ricci tensor has 10 independent components, for $n = 3$ it has 6, while for $n = 2$ there is only 1 because there is only one independent component of the Riemann curvature tensor to start off with.

There is one more contraction we can perform, namely on the Ricci tensor itself, to obtain what is called the *Ricci scalar* or *curvature scalar*

$$R := g^{\mu\nu} R_{\mu\nu} \quad . \tag{7.19}$$

One might have thought that in four dimensions there is another way of constructing a scalar, by contracting the Riemann tensor with the Levi-Civita tensor, but

$$\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = 0 \tag{7.20}$$

because of the Bianchi identity.

Note that for $n = 2$ the Riemann curvature tensor has as many independent components as the Ricci scalar, namely one, and that in three dimensions the Ricci tensor has as many components as the Riemann tensor, whereas in four dimensions there are strictly less components of the Ricci tensor than of the Riemann tensor. This has profound implications for the dynamics of gravity in these dimensions. In fact, we will see that it is only in dimensions $n > 3$ that gravity becomes truly dynamical, where empty space can be curved, where gravitational waves can exist etc.

7.5 An Example: The Curvature Tensor of the Two-Sphere

To see how all of this can be done in practice, let us work out the example of the two-sphere of unit radius. We already know the Christoffel symbols,

$$\Gamma^{\phi}{}_{\phi\theta} = \cot\theta \ , \qquad \Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta \ , \tag{7.21}$$

and we know that the Riemann curvature tensor has only one independent component. Let us therefore work out $R^{\theta}{}_{\phi\theta\phi}$. From the definition we find

$$R^{\theta}{}_{\phi\theta\phi} = \partial_{\theta}\Gamma^{\theta}{}_{\phi\phi} - \partial_{\phi}\Gamma^{\theta}{}_{\theta\phi} + \Gamma^{\theta}{}_{\theta\alpha}\Gamma^{\alpha}{}_{\phi\phi} - \Gamma^{\theta}{}_{\phi\alpha}\Gamma^{\alpha}{}_{\theta\phi} \ . \tag{7.22}$$

The second and third terms are manifestly zero, and we are left with

$$R^{\theta}{}_{\phi\theta\phi} = \partial_{\theta}(-\sin\theta\cos\theta) + \sin\theta\cos\theta\cot\theta = \sin^2\theta \ . \tag{7.23}$$

Thus we have

$$\begin{aligned} R^{\theta}{}_{\phi\theta\phi} &= R_{\theta\phi\theta\phi} = \sin^2\theta \\ R^{\phi}{}_{\theta\phi\theta} &= 1 \ . \end{aligned} \tag{7.24}$$

Therefore the Ricci tensor $R_{\mu\nu}$ has the components

$$\begin{aligned} R_{\theta\theta} &= 1 \\ R_{\theta\phi} &= 0 \\ R_{\phi\phi} &= \sin^2\theta \ . \end{aligned} \tag{7.25}$$

These equations can succinctly be written as

$$R_{\mu\nu} = g_{\mu\nu} \ , \tag{7.26}$$

sohwing that the standard metric on the two-sphere is what we will later call an *Einstein metric*. The Ricci scalar is

$$\begin{aligned} R &= g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} \\ &= 1 + \frac{1}{\sin^2\theta}\sin^2\theta \\ &= 2 \ . \end{aligned} \tag{7.27}$$

In particular, we have here our first concrete example of a space with non-trivial, in fact positive, curvature.

Question: what is the curvature scalar of a sphere of radius a?

Rather than redoing the calculation in that case, let us observe first of all that the Christoffel symbols are invariant under constant rescalings of the metric because they are schematically of the form $g^{-1}\partial g$. Therefore the Riemann curvature tensor, which only involves derivatives and products of Christoffel symbols, is also invariant. Hence the Ricci tensor, which is just a contraction of the Riemann tensor, is also invariant. However, to construct the Ricci scalar, one needs the inverse metric. This introduces an explicit a-dependence and the result is that the curvature scalar of a sphere of radius a is $R = 2/a^2$. In particular, the curvature scalar of a large sphere is smaller than that of a small sphere.

This result could also have been obtained on purely dimensional grounds. The curvature scalar is constructed from second derivatives of the metric. Hence it has length-dimension (-2). Therefore for a sphere of radius a, R has to be proportional to $1/a^2$. Comparing with the known result for $a = 1$ determines $R = 2/a^2$, as before.

7.6 * More on Curvature in 2 (spacelike) Dimensions

We can generalise the previous example somewhat, in this way connecting our considerations with the classical realm of the differential geometry of surfaces, in particular with the *Gauss Curvature* and with the *Liouville Equation*.

It is a simple exercise to derive the relation between the one independent component, say R_{1212}, of the Riemann tensor, and the scalar curvature. First of all, the Ricci tensor is

$$R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta} = R^{1}{}_{\alpha 1\beta} + R^{2}{}_{\alpha 2\beta} \tag{7.28}$$

so that the scalar cruvature is

$$R = g^{\alpha\beta} R_{\alpha\beta} = g^{11} R^{2}{}_{121} + g^{12} R^{1}{}_{112} + g^{21} R^{2}{}_{221} + g^{22} R^{1}{}_{212} \quad . \tag{7.29}$$

Using the fact that in 2 dimensions the components of the inverse metric are explicitly given by

$$\left(g^{\alpha\beta}\right) = \frac{1}{g_{11}g_{22} - g_{12}g_{21}} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{pmatrix} \tag{7.30}$$

and the (anti-)symmetry properties (1) and (2) of the Riemann tensor, one finds

$$R = \frac{2}{g_{11}g_{22} - g_{12}g_{21}} R_{1212} \quad . \tag{7.31}$$

The factor of 2 in this equation is a consequence of our (and the conventional) definition of the Riemann curvature tensor, and is responsible for the fact that the scalar curvature of the unit 2-sphere is $R = +2$. In two dimensions, it is more convenient and natural to absorb this factor of 2 into the definition of the (scalar) curvature, and what one then gets is the classical Gauss Curvature

$$K := \frac{1}{2} R \tag{7.32}$$

of a two-dimensional surface.

When we specialise the above to the class of *conformally flat* metrics with line element

$$ds^2 = \mathrm{e}^{2h(x,y)}(dx^2 + dy^2) \quad \Leftrightarrow \quad g_{\alpha\beta} = \exp 2h(x,y)\delta_{\alpha\beta} \tag{7.33}$$

one finds the simple (and easy to memorise) results

$$R^x_{\ yxy} = -\Delta h \tag{7.34}$$

and

$$K = -\mathrm{e}^{-2h}\Delta h \tag{7.35}$$

where $\Delta = \partial_x^2 + \partial_y^2$ is the 2-dimensional Laplacian with respect to the flat Euclidean metric $dx^2 + dy^2$. Thus a surface with *constant curvature* $K = k$ is given by a solution to the non-linear differential equation

$$\Delta h + k\mathrm{e}^{2h} = 0 \ . \tag{7.36}$$

This is the (in-)famous *Liouville equation*, which plays a fundamental role in many branches of mathematics (and mathematical physics). In terms of the intrinsic Laplacian Δ_g associated to the metric $g_{\alpha\beta}$, the Gaussian cuvature and the Liouville equation can also simply be written as

$$K = -\Delta_g h \qquad , \qquad \Delta_g h + k = 0 \ , \tag{7.37}$$

since, due to the peculiarities of 2 dimensions, $\sqrt{g}g^{\alpha\beta}$ in independent of h, i.e. is conformally invariant (as we already observed in a different context in section 5.5, cf. (5.39)),

$$\begin{aligned} \sqrt{g}g^{\alpha\beta} &= \mathrm{e}^{2h}\mathrm{e}^{-2h}\delta^{\alpha\beta} = \delta^{\alpha\beta} \\ \Rightarrow \quad \Delta_g &= \frac{1}{\sqrt{g}}\partial_\alpha(\sqrt{g}g^{\alpha\beta}\partial_\beta) = \frac{1}{\sqrt{g}}\partial_\alpha(\delta^{\alpha\beta}\partial_\beta) = \mathrm{e}^{-2h}\Delta \ . \end{aligned} \tag{7.38}$$

I will not attempt to say anything about the general (local) solution of this equation (which roughly speaking depends on an arbitrary meromorphic function of the complex coordinate $z = x + iy$), but close this section with some special (and particularly prominent) solutions of this equation.

1. It is easy to see that

   $$\mathrm{e}^{2h(x,y)} = y^{-2} \quad \Leftrightarrow \quad h(x,y) = -\ln y \tag{7.39}$$

   solves the Liouville equation with $k = -1$. The corresponding space of constant negative curvature is the *Poincaré upper-half plane* model of the hyperbolic geometry,

   $$ds^2 = \frac{dx^2 + dy^2}{y^2} \qquad (\ (x,y) \in \mathbb{R}^2,\ y > 0\ ) \ . \tag{7.40}$$

   By the coordinate transformation $y = e^z$ this is mapped to the equivalent metric

   $$ds^2 = dz^2 + \mathrm{e}^{-2z}dx^2 \tag{7.41}$$

   on the entire (x,z)-plane.

2. Another solution (for any k) is the rotationally invariant function

$$e^{2h(x,y)} = 4(1 + k(x^2 + y^2))^{-2} \quad \Leftrightarrow \quad h = -\ln(1 + k(x^2 + y^2)) + \text{const.} \tag{7.42}$$

   (a) For $k = 0$ one finds the flat (zero curvature) Euclidean metric on $\mathbb{R}^2$.

   (b) For $k = +1$, one obtains the metric

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 + x^2 + y^2)^2} \quad . \tag{7.43}$$

   This is the constant positive curvature metric on the *Riemann sphere* one gets by stereographic projection of the standard metric on the two-sphere S^2 to the (x, y)-plane.

   (c) For $k = -1$, one finds

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2} \qquad (\ \{x, y\} \in \mathbb{R}^2, x^2 + y^2 < 1\ ) \ . \tag{7.44}$$

   This is the *Poincaré disc* model of the hyperbolic geometry, defined in the interior of the unit disc in $\mathbb{R}^2$. The two metrics (7.40) and (7.44) are isometric, i.e. related by a (not completely evident) coordinate transformation.

It is worth remarking that the Poincaré upper-half plane model of a space with constant negative curvature readily generalises to arbitrary dimension and signature. Thus

$$ds^2 = \frac{d\vec{x}^2 + dy^2}{y^2} \quad , \quad d\vec{x}^2 = \delta_{ab} dx^a dx^b \quad \text{or} \quad d\vec{x}^2 = \eta_{ab} dx^a dx^b \tag{7.45}$$

is the metric of a $(d + 1)$-dimensional space(-time) with constant negative curvature.

The Lorentzian metric will reappear later as a solution to the Einstein equations with a negative cosmological constant, and is in this context known as anti-de Sitter metric (in Poincaré coordinates, which cover only a part of the complete space-time).

7.7 Bianchi Identities

So far, we have discussed algebraic properties of the Riemann tensor. But the Riemann tensor also satisfies some differential identities which, in particular in their contracted form, will be of fundamental importance in the following.

The first identity is easy to derive. As a (differential) operator the covariant derivative clearly satisfies the Jacobi identity

$$[\nabla_{[\mu}, [\nabla_\nu, \nabla_{\lambda]}]] = 0 \tag{7.46}$$

If you do not believe this, just write out the twelve relevant terms explicitly to see that this identity is true:

$$\begin{aligned}
[\nabla_{[\mu},[\nabla_\nu,\nabla_{\lambda]}]] &\sim \nabla_\mu\nabla_\nu\nabla_\lambda - \nabla_\mu\nabla_\lambda\nabla_\nu - \nabla_\nu\nabla_\lambda\nabla_\mu + \nabla_\lambda\nabla_\nu\nabla_\mu \\
&+ \nabla_\lambda\nabla_\mu\nabla_\nu - \nabla_\lambda\nabla_\nu\nabla_\mu + \nabla_\nu\nabla_\mu\nabla_\lambda - \nabla_\mu\nabla_\nu\nabla_\lambda \\
&+ \nabla_\nu\nabla_\lambda\nabla_\mu - \nabla_\nu\nabla_\mu\nabla_\lambda - \nabla_\lambda\nabla_\mu\nabla_\nu + \nabla_\mu\nabla_\lambda\nabla_\nu \\
&= 0 \;. \qquad (7.47)
\end{aligned}$$

Hence, recalling the definition of the curvature tensor in terms of commutators of covariant derivatives, we obtain

$$\begin{aligned}
\text{Jacobi Identity} \quad \Rightarrow \quad \text{Bianchi identity:} \quad & R_{\alpha\beta[\mu\nu;\lambda]} = 0 \\
\Leftrightarrow \quad & \nabla_{[\lambda}R_{\alpha\beta]\mu\nu} = 0 \;. \qquad (7.48)
\end{aligned}$$

Because of the antisymmetry of the Riemann tensor in the first two indices, this can also be written more explicitly as

$$\nabla_\lambda R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\lambda\mu} + \nabla_\mu R_{\alpha\beta\nu\lambda} = 0 \;. \qquad (7.49)$$

By contracting this with $g^{\alpha\mu}$ we obtain

$$\nabla_\lambda R_{\beta\nu} - \nabla_\nu R_{\beta\lambda} + \nabla_\mu R^\mu_{\ \beta\nu\lambda} = 0 \;. \qquad (7.50)$$

To also turn the last term into a Ricci tensor we contract once more, with $g^{\beta\lambda}$ to obtain the contracted Bianchi identity

$$\nabla_\lambda R^\lambda_{\ \nu} - \nabla_\nu R + \nabla_\mu R^\mu_{\ \nu} = 0 \;, \qquad (7.51)$$

or

$$\nabla^\mu(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R) = 0 \;. \qquad (7.52)$$

The tensor appearing in this equation is the so-called *Einstein tensor* $G_{\mu\nu}$,

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R \;. \qquad (7.53)$$

It is the unique divergence-free tensor that can be built from the metric and its first and second derivatives (apart from $g_{\mu\nu}$ itself, of course), and this is why it will play *the* central role in the Einstein equations for the gravitational field.

7.8 Another Look at the Principle of General Covariance

In the section on the principle of minimal coupling, I mentioned that this algorithm or the principle of general covariance do not necessarily fix the equations uniquely. In other words, there could be more than one generally covariant equation which reduces to a given equation in Minkowski space. Having the curvature tensor at our disposal now, we can construct examples of this kind.

As a first example, consider a massive particle with spin, characterised by a spin tensor $S^{\mu\nu}$. We could imagine the possibility that in a gravitiational field there is a coupling between the spin and the curvature, so that the particle does not follow a geodesic, but rather obeys an equation of the (Mathisson-Papapetrou) type

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda + aR^\mu_{\nu\lambda\rho}\dot{x}^\nu S^{\lambda\rho} = 0 \quad . \tag{7.54}$$

This equation is clearly tensorial (generally covariant) and reduces to the equation for a straight line in Minkowski space, but differs from the geodesic equation (which has the same properties) for $a \neq 0$. But, since the Riemann tensor is second order in derivatives, a has to be a dimensionful quantity (of length dimension 1) for this equation to make sense. Thus the rationale for usually not considering such additional terms is that they are irrelevant at scales large compared to some characteristic size of the particle, say its Compton wave length.

We will mostly be dealing with weak gravitational fields and other low-energy phenomena and under those circumstances the minimal coupling rule can be trusted. However, it is not ruled out that under extreme conditions (very strong or strongly fluctuating gravitational fields) such terms are actually present and relevant.

For another example, consider the wave equation for a (massless, say) scalar field Φ. In Minkowski space, this is the Klein-Gordon equation which has the obvious curved space analogue (4.48)

$$\Box\Phi = 0 \tag{7.55}$$

obtained by the minimal coupling description. However, one could equally well postulate the equation

$$(\Box + aR)\Phi = 0 \quad , \tag{7.56}$$

where a is a (dimensionless) constant and R is the scalar curvature. This equation is generally covariant, and reduces to the ordinary Klein-Gordon equation in Minkowski space, so this is an acceptable curved-space extension of the wave equation for a scalar field. Moreover, since here a is dimensionless, we cannot argue as above that this ambiguity is irrelevant for weak fields. Indeed, one frequently postulates a specific non-zero value for a which makes the wave equation *conformally invariant* (invariant under position-dependent rescalings of the metric) for massless fields. This is an ambiguity we have to live with.

8 Curvature II: Geometry and Curvature

In this section, we will discuss three properties of the Riemann curvature tensor that illustrate its geometric significance and thus, *a posteriori*, justify equating the commutator of covariant derivatives with the intuitive concept of curvature. These properties are

- the path-dependence of parallel transport in the presence of curvature,
- the fact that the space-time metric is equivalent to the (in an obvious sense flat) Minkowski metric if and only if the Riemann curvature tensor vanishes, and
- the geodesic deviation equation describing the effect of curvature on the trajectories of families of freely falling particles.

8.1 Intrinsic Geometry, Curvature and Parallel Transport

The Riemann curvature tensor and its relatives, introduced above, measure the intrinsic geometry and curvature of a space or space-time. This means that they can be calculated by making experiments and measurements on the space itself. Such experiments might involve things like checking if the interior angles of a triangle add up to π or not.

An even better method, the subject of this section, is to check the properties of parallel transport. The tell-tale sign (or smoking gun) of the presence of curvature is the fact that parallel transport is path dependent, i.e. that parallel transporting a vector V from a point A to a point B along two different paths will in general produce two different vectors at B. Another way of saying this is that parallel transporting a vector around a closed loop at A will in general produce a new vector at A which differs from the initial vector.

This is easy to see in the case of the two-sphere (see Figure 8). Since all the great circles on a two-sphere are geodesics, in particular the segments N-C, N-E, and E-C in the figure, we know that in order to parallel transport a vector along such a line we just need to make sure that its length and the angle between the vector and the geodesic line are constant. Thus imagine a vector 1 at the north pole N, pointing downwards along the line N-C-S. First parallel transport this along N-C to the point C. There we will obtain the vector 2, pointing downwards along C-S. Alternatively imagine parallel transporting the vector 1 first to the point E. Since the vector has to remain at a constant (right) angle to the line N-E, at the point E parallel transport will produce the vector 3 pointing westwards along E-C. Now parallel transporting this vector along E-C to C will produce the vector 4 at C. This vector clearly differs from the vector 2 that was obtained by parallel transporting along N-C instead of N-E-C.

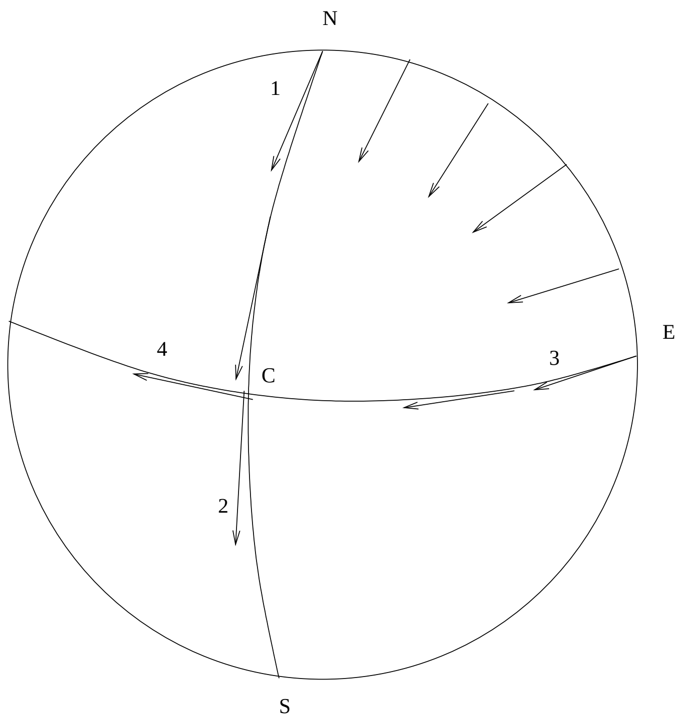

Figure 8: Figure illustrating the path dependence of parallel transport on a curved space: Vector 1 at N can be parallel transported along the geodesic N-S to C, giving rise to Vector 2. Alternatively, it can first be transported along the geodesic N-E (Vector 3) and then along E-C to give the Vector 4. Clearly these two are different. The angle between them reflects the curvature of the two-sphere.

To illustrate the claim about closed loops above, imagine parallel transporting vector 1 along the closed loop N-E-C-N from N to N. In order to complete this loop, we still have to parallel transport vector 4 back up to N. Clearly this will give a vector, not indicated in the figure, different from (and pointing roughly at a right angle to) the vector 1 we started off with.

This intrinsic geometry and curvature described above should be contrasted with the extrinsic geometry which depends on how the space may be embedded in some larger space.

For example, a cylinder can be obtained by 'rolling up' $\mathbb{R}^2$. It clearly inherits the flat metric from $\mathbb{R}^2$ and if you calculate its curvature tensor you will find that it is zero. Thus, the intrinsic curvature of the cylinder is zero, and the fact that it looks curved to an outside observer is not something that can be detected by somebody living on the cylinder. For example, parallel transport is rather obviously path independent.

As we have no intention of embedding space-time into something higher dimensional, we will only be concerned with intrinsic geometry in the following. However, if you would for example be interested in the properties of spacelike hypersurfaces in space-time, then aspects of both intrinsic and extrinsic geometry of that hypersurface would be relevant.

The precise statement regarding the relation between the path dependence of parallel transport and the presence of curvature is the following. If one parallel transports a covector V_μ (I use a covector instead of a vector only to save myself a few minus signs here and there) along a closed infinitesimal loop $x^\mu(\tau)$ with, say, $x(\tau_0) = x(\tau_1) = x_0$, then one has

$$V_\mu(\tau_1) - V_\mu(\tau_0) = \tfrac{1}{2}(\oint x^\rho dx^\nu) R^\sigma_{\ \mu\rho\nu}(x_0) V_\sigma(\tau_0) \ . \tag{8.1}$$

Thus an arbitrary vector V^μ will not change under parallel transport around an arbitrary small loop at x_0 only if the curvature tensor at x_0 is zero. This can of course be extended to finite loops, but the important point is that in order to detect curvature at a given point one only requires parallel transport along infinitesimal loops.

Before turning to a proof of this result, I just want to note that intuitively it can be understood directly from the definition of the curvature tensor (7.2). Imagine that the infinitesimal loop is actually a tiny parallelogram made up of the coordinate lines x^1 and x^2. Parallel transport along x^1 is governed by the equation $\nabla_1 V^\mu = 0$, that along x^2 by $\nabla_2 V^\mu = 0$. The fact that parallel transporting first along x^1 and then along x^2 can be different from doing it the other way around is precisely the statement that ∇_1 and ∇_2 do not commute, i.e. that some of the components $R_{\mu\nu 12}$ of the curvature tensor are non-zero.

To establish (8.1) we first reformulate the condition of parallel transport,

$$\frac{D}{D\tau} V_\mu = 0 \quad \Leftrightarrow \frac{d}{d\tau} V_\mu = \Gamma^\lambda_{\mu\nu} \dot{x}^\nu V_\lambda \tag{8.2}$$

with the initial condition at $\tau = \tau_0$ as the integral equation

$$V_\mu(\tau) = V_\mu(\tau_0) + \int_{\tau_0}^{\tau} d\tau'\ \Gamma^\lambda_{\ \mu\nu}(x(\tau'))\dot{x}^\nu(\tau')V_\lambda(\tau')\ \ . \tag{8.3}$$

As usual, such an equation can be 'solved' by iteration (leading to a time-ordered exponential). Keeping only the first two non-trivial terms in the iteration, one has

$$\begin{aligned} V_\mu(\tau) \ &= \ V_\mu(\tau_0) + \int_{\tau_0}^{\tau} d\tau'\ \Gamma^\lambda_{\ \mu\nu}(x(\tau'))\dot{x}^\nu(\tau')V_\lambda(\tau_0) \\ &\quad + \int_{\tau_0}^{\tau} d\tau' \int_{\tau_0}^{\tau'} d\tau''\ \Gamma^\lambda_{\ \mu\nu}(x(\tau'))\dot{x}^\nu(\tau')\Gamma^\sigma_{\ \lambda\rho}(x(\tau''))\dot{x}^\rho(\tau'')V_\sigma(\tau_0) \\ &\quad + \ldots \end{aligned} \tag{8.4}$$

For sufficiently small (infinitesimal) loops, we can expand the Christoffel symbols as

$$\Gamma^\lambda_{\ \mu\nu}(x(\tau)) = \Gamma^\lambda_{\ \mu\nu}(x_0) + (x(\tau) - x_0)^\rho(\partial_\rho\Gamma^\lambda_{\ \mu\nu})(x_0) + \ldots \tag{8.5}$$

The linear term in the expansion of $V_\mu(\tau)$ arises from the zero'th order contribution $\Gamma^\lambda_{\ \mu\nu}(x_0)$ in the first order (single integral) term in (8.4),

$$[V_\mu(\tau_1) - V_\mu(\tau_0)]^{(1)} = \Gamma^\lambda_{\ \mu\nu}(x_0)V_\lambda(\tau_0)(\int_{\tau_0}^{\tau_1} d\tau'\ \dot{x}^\nu(\tau'))\ \ . \tag{8.6}$$

Now the important observation is that, for a closed loop, the integral in brackets is zero,

$$\int_{\tau_0}^{\tau_1} d\tau'\dot{x}^\nu(\tau') = x^\nu(\tau_1) - x^\nu(\tau_0) = 0\ \ . \tag{8.7}$$

Thus the change in $V_\mu(\tau)$, when transported along a small loop, is at least of second order. Such second order terms arise in two different ways, from the first order term in the expansion of $\Gamma^\lambda_{\ \mu\nu}(x)$ in the first order term in (8.4), and from the zero'th order terms $\Gamma^\lambda_{\ \mu\nu}(x_0)$ in the quadratic (double integral) term in (8.4),

$$\begin{aligned} [V_\mu(\tau_1) - V_\mu(\tau_0)]^{(2)} \ &= \ (\partial_\rho\Gamma^\lambda_{\ \mu\nu})(x_0)V_\lambda(\tau_0)(\int_{\tau_0}^{\tau_1} d\tau'\ (x(\tau') - x_0)^\rho\dot{x}^\nu(\tau')) \\ &\quad + \ (\Gamma^\lambda_{\ \mu\nu}\Gamma^\sigma_{\ \lambda\rho})(x_0)V_\sigma(\tau_0)\int_{\tau_0}^{\tau_1} d\tau' \int_{\tau_0}^{\tau'} d\tau''\ \dot{x}^\nu(\tau')\dot{x}^\rho(\tau'') \end{aligned} \tag{8.8}$$

The τ''-integral can be performed explicitly,

$$\int_{\tau_0}^{\tau_1} d\tau' \int_{\tau_0}^{\tau'} d\tau''\ \dot{x}^\nu(\tau')\dot{x}^\rho(\tau'') = \int_{\tau_0}^{\tau_1} d\tau'\ \dot{x}^\nu(\tau')(x(\tau') - x_0)^\rho = \int_{\tau_0}^{\tau_1} d\tau'\ \dot{x}^\nu(\tau')x^\rho(\tau') \tag{8.9}$$

and therefore we find

$$V_\mu(\tau_1) - V_\mu(\tau_0) \approx (\partial_\rho\Gamma^\sigma_{\ \mu\nu} + \Gamma^\lambda_{\ \mu\nu}\Gamma^\sigma_{\ \lambda\rho})(x_0)V_\sigma(\tau_0)(\int_{\tau_0}^{\tau_1} d\tau'\ \dot{x}^\nu(\tau')x^\rho(\tau')) \tag{8.10}$$

The final observation we need is that the remaining integral is anti-symmetric in the indices ν, ρ, which follows immediately from

$$\int_{\tau_0}^{\tau_1} d\tau' \, (\dot{x}^\nu(\tau')x^\rho(\tau') + x^\nu(\tau')\dot{x}^\rho(\tau') = \int_{\tau_0}^{\tau_1} d\tau' \, \frac{d}{d\tau'}(x^\nu(\tau')x^\rho(\tau')) = 0 \ . \tag{8.11}$$

It now follows from (8.10) and the definition of the Riemann tensor that

$$V_\mu(\tau_1) - V_\mu(\tau_0) = \tfrac{1}{2}(\oint x^\rho dx^\nu) R^\sigma_{\ \mu\rho\nu}(x_0) V_\sigma(\tau_0) \ . \tag{8.12}$$

8.2 Vanishing Riemann Tensor and Existence of Flat Coordinates

We are now finally in a position to prove the converse to the statement that the Minkowski metric has vanishing Riemann tensor. Namely, we will see that when the Riemann tensor of a metric vanishes, there are coordinates in which the metric is the standard Minkowski metric. Since the opposite of curved is flat, this then allows one to unambiguously refer to the Minkowski metric as *the* flat metric, and to Minkowski space as *flat space(-time)*.

So let us assume that we are given a metric with vanishing Riemann tensor. Then, by the above, parallel transport is path independent and we can, in particular, extend a vector $V^\mu(x_0)$ to a vector field everywhere in space-time: to define $V^\mu(x_1)$ we choose any path from x_0 to x_1 and use parallel transport along that path. In particular, the vector field V^μ, defined in this way, will be covariantly constant or *parallel*, $\nabla_\mu V^\nu = 0$. We can also do this for four linearly independent vectors V_a^μ at x_0 and obtain four covariantly constant (parallel) vector fields which are linearly independent at every point.

An alternative way of saying or seeing this is the following: The integrability condition for the equation $\nabla_\mu V^\lambda = 0$ is

$$\nabla_\mu V^\lambda = 0 \ \Rightarrow \ [\nabla_\mu, \nabla_\nu] V^\lambda = R^\lambda_{\ \sigma\mu\nu} V^\sigma = 0 \ . \tag{8.13}$$

This means that the (4×4) matrices $M(\mu, \nu)$ with coefficients $M(\mu,\nu)^\lambda_{\ \sigma} = R^\lambda_{\ \sigma\mu\nu}$ have a zero eigenvalue. If this integrability condition is satisfied, a solution to $\nabla_\mu V^\lambda$ can be found. If one wants four linearly independent parallel vector fields, then the matrices $M(\mu, \nu)$ must have four zero eigenvalues, i.e. they are zero and therefore $R^\lambda_{\ \sigma\mu\nu} = 0$. If this condition is satisfied, all the integrability conditions are satisfied and there will be four linearly independent covariantly constant vector fields - the same conclusion as above.

We will now use this result in the proof, but for covectors instead of vectors. Clearly this makes no difference: if V^μ is a parallel vector field, then $g_{\mu\nu}V^\nu$ is a parallel covector field.

Fix some point x_0. At x_0, there will be an invertible matrix e^a_μ such that

$$g^{\mu\nu}(x_0) e^a_\mu e^b_\nu = \eta^{ab} \ . \tag{8.14}$$

Now we solve the equations

$$\nabla_\nu E^a_\mu = 0 \Leftrightarrow \partial_\nu E^a_\mu = \Gamma^\lambda_{\ \mu\nu} E^a_\lambda \tag{8.15}$$

with the initial condition $E^a_\mu(x_0) = e^a_\mu$. This gives rise to four linearly independent parallel covectors E^a_μ.

Now it follows from (8.15) that

$$\partial_\mu E^a_\nu = \partial_\nu E^a_\mu \ . \tag{8.16}$$

Therefore locally there are four scalars ξ^a such that

$$E^a_\mu = \frac{\partial \xi^a}{\partial x^\mu} \ . \tag{8.17}$$

These are already the flat coordinates we have been looking for. To see this, consider the expression $g^{\mu\nu} E^a_\mu E^b_\nu$. This is clearly constant because the metric and the E^a_μ are covariantly constant,

$$\partial_\lambda (g^{\mu\nu} E^a_\mu E^b_\nu) = \nabla_\lambda (g^{\mu\nu} E^a_\mu E^b_\nu) = 0 \ . \tag{8.18}$$

But at x_0, this is just the flat metric and thus

$$(g^{\mu\nu} E^a_\mu E^b_\nu)(x) = (g^{\mu\nu} E^a_\mu E^b_\nu)(x_0) = \eta^{ab} \ . \tag{8.19}$$

Summing this up, we have seen that, starting from the assumption that the Riemann curvature tensor of a metric $g_{\mu\nu}$ is zero, we have proven the existence of coordinates ξ^a in which the metric takes the Minkowski form,

$$g_{\mu\nu} = \frac{\partial \xi^a}{\partial x^\mu} \frac{\partial \xi^b}{\partial x^\nu} \eta_{ab} \ . \tag{8.20}$$

8.3 The Geodesic Deviation Equation

In a certain sense the main effect of curvature (or gravity) is that initially parallel trajectories of freely falling non-interacting particles (dust, pebbles,...) do not remain parallel, i.e. that gravity has the tendency to focus (or defocus) matter. This statement find its mathematically precise formulation in the geodesic deviation equation.

Let us, as we will need this later anyway, recall first the situation in the Newtonian theory. One particle moving under the influence of a gravitational field is governed by the equation

$$\tfrac{d^2}{dt^2} x^i = -\partial^i \phi(x) \ , \tag{8.21}$$

where ϕ is the potential. Now consider a family of particles, or just two nearby particles, one at $x^i(t)$ and the other at $x^i(t) + \delta x^i(t)$. The other particle will of course obey the equation

$$\tfrac{d^2}{dt^2}(x^i + \delta x^i) = -\partial^i \phi(x + \delta x) \ . \tag{8.22}$$

From these two equations one can deduce an equation for δx itself, namely

$$\tfrac{d^2}{dt^2}\delta x^i = -\partial^i\partial_j\phi(x)\delta x^j \quad . \tag{8.23}$$

It is the counterpart of this equation that we will be seeking in the context of General Relativity. The starting point is of course the geodesic equation for x^μ and for its nearby partner $x^\mu + \delta x^\mu$,

$$\tfrac{d^2}{d\tau^2}x^\mu + \Gamma^\mu_{\ \nu\lambda}(x)\tfrac{d}{d\tau}x^\nu\tfrac{d}{d\tau}x^\lambda = 0 \quad , \tag{8.24}$$

and

$$\tfrac{d^2}{d\tau^2}(x^\mu + \delta x^\mu) + \Gamma^\mu_{\ \nu\lambda}(x+\delta x)\tfrac{d}{d\tau}(x^\nu + \delta x^\nu)\tfrac{d}{d\tau}(x^\lambda + \delta x^\lambda) = 0 \quad . \tag{8.25}$$

As above, from these one can deduce an equation for δx, namely

$$\tfrac{d^2}{d\tau^2}\delta x^\mu + 2\Gamma^\mu_{\ \nu\lambda}(x)\tfrac{d}{d\tau}x^\nu\tfrac{d}{d\tau}\delta x^\lambda + \partial_\rho\Gamma^\mu_{\ \nu\lambda}(x)\delta x^\rho\tfrac{d}{d\tau}x^\nu\tfrac{d}{d\tau}x^\lambda = 0 \quad . \tag{8.26}$$

Now this does not look particularly covariant. Thus instead of in terms of $d/d\tau$ we would like to rewrite this in terms of covariant operator $D/D\tau$, with

$$\frac{D}{D\tau}\delta x^\mu = \frac{d}{d\tau}\delta x^\mu + \Gamma^\mu_{\ \nu\lambda}\frac{dx^\nu}{d\tau}\delta x^\lambda \quad . \tag{8.27}$$

Calculating $(D/D\tau)^2\delta x^\mu$, replacing $\ddot{x}^\mu$ appearing in that expression by $-\Gamma^\mu_{\ \nu\lambda}\dot{x}^\nu\dot{x}^\lambda$ (because x^μ satisfies the geodesic equation) and using (8.26), one finds the nice covariant *geodesic deviation equation*

$$\frac{D^2}{D\tau^2}\delta x^\mu = R^\mu_{\ \nu\lambda\rho}\dot{x}^\nu\dot{x}^\lambda\delta x^\rho \quad . \tag{8.28}$$

Note that for flat space(-time), this equation reduces to

$$\frac{d^2}{d\tau^2}\delta x^\mu = 0 \quad , \tag{8.29}$$

which has the solution

$$\delta x^\mu = A^\mu\tau + B^\mu \quad . \tag{8.30}$$

In particular, one recovers Euclid's parallel axiom that two straight lines intersect at most once and that when they are initially parallel they never intersect. This shows very clearly that intrinsic curvature leads to non-Euclidean geometry in which e.g. the parallel axiom is not necessarily satisifed.

It is also possible to give a manifestly covariant, and thus perhaps slightly more satisfactory, derivation of the above gedesic deviation equation. The starting point is a geodesic vector field u^μ, $u^\nu\nabla_\nu u^\mu = 0$, and a deviation vector field $\delta x^\mu = \xi^\mu$ characterised by the condition

$$[u,\xi]^\mu = u^\nu\nabla_\nu\xi^\mu - \xi^\nu\nabla_\nu u^\mu = 0 \quad . \tag{8.31}$$

The rationale for this condition is that, if $x^\mu(\tau, s)$ is a family of geodesics labelled by s, one has the identifications

$$u^\mu = \frac{\partial}{\partial\tau}x^\mu(\tau,s) \quad , \quad \xi^\mu = \frac{\partial}{\partial s}x^\mu(\tau,s) \ . \tag{8.32}$$

Since second partial derivatives commute, this implies the relation

$$\frac{\partial}{\partial\tau}\xi^\mu(\tau,s) = \frac{\partial}{\partial s}u^\mu(\tau,s) \ , \tag{8.33}$$

(implicit in the identification $\delta\dot{x} = (d/d\tau)\delta x$ employed in the above derivation). The condition (8.31) is nothing other than the covariant way of writing (8.33).

Given this set-up, we now want to calculate

$$\frac{D^2}{D\tau^2}\xi^\mu = u^\lambda\nabla_\lambda(u^\nu\nabla_\nu\xi^\mu) \ . \tag{8.34}$$

Using (8.31) twice, one finds

$$\begin{aligned} u^\lambda\nabla_\lambda(u^\nu\nabla_\nu\xi^\mu) &= u^\lambda\nabla_\lambda(\xi^\nu\nabla_\nu u^\mu) \\ &= (u^\lambda\nabla_\lambda\xi^\nu)\nabla_\nu u^\mu + u^\lambda\xi^\nu\nabla_\lambda\nabla_\nu u^\mu \\ &= (\xi^\lambda\nabla_\lambda u^\nu)\nabla_\nu u^\mu + u^\lambda\xi^\nu[\nabla_\lambda,\nabla_\nu]u^\mu + u^\lambda\xi^\nu\nabla_\nu\nabla_\lambda u^\mu \ . \end{aligned} \tag{8.35}$$

Rewriting the last term as

$$u^\lambda\xi^\nu\nabla_\nu\nabla_\lambda u^\mu = \xi^\nu\nabla_\nu(u^\lambda\nabla_\lambda u^\mu) - (\xi^\nu\nabla_\nu u^\lambda)\nabla_\lambda u^\mu \tag{8.36}$$

one sees that the first term of the last line of (8.35) cancels and, using the definition of the curvature tensor, one is left with

$$u^\lambda\nabla_\lambda(u^\nu\nabla_\nu\xi^\mu) = R^\mu_{\ \sigma\lambda\nu}u^\sigma u^\lambda\xi^\nu + \xi^\nu\nabla_\nu(u^\lambda\nabla_\lambda u^\mu) \ . \tag{8.37}$$

So far, we have only used the condition $[u,\xi]^\mu = 0$ (8.31). Thus, for a general family of curves (the integral curves of u^μ), the deviation vector ξ^μ feels a force which is due to both the curvature of space-time and the acceleration $a^\mu = u^\lambda\nabla_\lambda u^\mu$ of the family of curves. For geodesics, the latter is absent and one finds the geodesic deviation equation (8.28) in the form

$$u^\lambda\nabla_\lambda(u^\nu\nabla_\nu\xi^\mu) = R^\mu_{\ \sigma\lambda\nu}u^\sigma u^\lambda\xi^\nu \ . \tag{8.38}$$

8.4 * The Raychaudhuri Equation

Manipulations similar to those leading from (8.35) to (8.38) allow one to derive an equation for the rate of change of the divergence $\nabla_\mu u^\mu$ of a family of geodesics along the geodesics. This simple result, known as the *Raychaudhuri equation*, has important implications and ramifications in general relativity, in particular in the context of the so-called *singularity theorems* of Penrose, Hawking and others, none of which will, however, be explored here.

To set the stage, let V^μ be an, at first, arbitrary vector field. From the definition of the curvature tensor,

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)V^\lambda = R^\lambda_{\ \rho\mu\nu}V^\rho \quad , \tag{8.39}$$

one deduces, after contracting the indices μ and λ,

$$(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)V^\mu = R_{\mu\nu}V^\mu \quad . \tag{8.40}$$

Multiplying by V^ν, one finds

$$V^\nu\nabla_\mu\nabla_\nu V^\mu - V^\nu\nabla_\nu\nabla_\mu V^\mu = R_{\mu\nu}V^\mu V^\nu \quad . \tag{8.41}$$

Rewriting the first term as

$$V^\nu\nabla_\mu\nabla_\nu V^\mu = \nabla_\mu(V^\nu\nabla_\nu V^\mu) - (\nabla_\mu V^\nu)(\nabla_\nu V^\mu) \tag{8.42}$$

this identity can be written as

$$V^\nu\nabla_\nu(\nabla_\mu V^\mu) + (\nabla_\mu V_\nu)(\nabla^\nu V^\mu) - \nabla_\mu(V^\nu\nabla_\nu V^\mu) + R_{\mu\nu}V^\mu V^\nu = 0 \quad . \tag{8.43}$$

Now apply this identity to a geodesic vector field $V^\mu \to u^\mu$, $u^\nu\nabla_\nu u^\mu = 0$. Then the third term disappears and one has

$$u^\nu\nabla_\nu(\nabla_\mu u^\mu) + (\nabla_\mu u_\nu)(\nabla^\nu u^\mu) + R_{\mu\nu}u^\mu u^\nu = 0 \quad . \tag{8.44}$$

Here the first term is the rate of change of the divergence

$$\theta = \nabla_\mu u^\mu \tag{8.45}$$

along u^μ,

$$u^\nu\nabla_\nu(\nabla_\mu u^\mu) = \frac{d}{d\tau}\theta \quad , \tag{8.46}$$

and we can therefore write the above result as

$$\frac{d}{d\tau}\theta = -(\nabla_\mu u_\nu)(\nabla^\nu u^\mu) - R_{\mu\nu}u^\mu u^\nu \quad . \tag{8.47}$$

To gain some more insight into the geometric significance of this equation, consider the case that u^μ is timelike and normalised as $u^\mu u_\mu = -1$ (so that τ is proper time) and introduce the tensor

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \quad . \tag{8.48}$$

It has the characteristic property that it is orthogonal to u^μ,

$$u^\mu h_{\mu\nu} = h_{\mu\nu}u^\nu = 0 \quad . \tag{8.49}$$

It can therefore be interpreted as the spatial projection of the metric in the directions orthogonal to the timelike vector field u^μ. This can be seen more explicitly in terms of the projectors

$$\begin{aligned} p^\mu_{\ \nu} &= \delta^\mu_{\ \nu} + u^\mu u_\nu \\ p^\mu_{\ \nu}p^\nu_{\ \lambda} &= p^\mu_{\ \lambda} \quad . \end{aligned} \tag{8.50}$$

On directions tangential to u^μ they act as

$$p^\mu_{\ \nu} u^\nu = 0 \ \ , \tag{8.51}$$

whereas on vectors ξ^μ orthogonal to u^μ, $u_\mu \xi^\mu = 0$ (spacelike vectors), one has

$$p^\mu_{\ \nu} \xi^\nu = \xi^\mu \ \ . \tag{8.52}$$

Thus, acting on an arbitrary vector field V^μ, $p^\mu_{\ \nu} V^\nu$ is the projection of this vector into the plane orthogonal to u^μ. In the same way one can project an arbitrary tensor. For example, the projection of the metric is

$$g_{\mu\nu} \rightarrow g_{\lambda\rho} p^\lambda_{\ \mu} p^\rho_{\ \nu} = g_{\mu\nu} + u_\mu u_\nu = h_{\mu\nu} \ \ , \tag{8.53}$$

as anticipated above. In particular, while for the space-time metric one obviously has $g^{\mu\nu} g_{\mu\nu} = 4$, the trace of $h_{\mu\nu}$ is

$$g^{\mu\nu} h_{\mu\nu} = g^{\mu\nu} g_{\mu\nu} + g^{\mu\nu} u_\mu u_\nu = 4 - 1 = 3 = h^{\mu\nu} h_{\mu\nu} \ \ . \tag{8.54}$$

Now let us introduce the tensor

$$B_{\mu\nu} = \nabla_\nu u_\mu \ \ . \tag{8.55}$$

This tensor satisfies

$$u^\mu B_{\mu\nu} = \tfrac{1}{2} \nabla_\nu (u_\mu u^\mu) = \tfrac{1}{2} \nabla_\nu (-1) = 0 \tag{8.56}$$

and

$$B_{\mu\nu} u^\nu = u^\nu \nabla_\nu u_\mu = 0 \tag{8.57}$$

and is thus a spatial tensor in the sense above. We now decompose $B_{\mu\nu}$ into its anti-symmetric, symmetric-traceless and trace part,

$$B_{\mu\nu} = \omega_{\mu\nu} + \sigma_{\mu\nu} + \tfrac{1}{3} \theta h_{\mu\nu} \ \ , \tag{8.58}$$

with

$$\begin{aligned} \omega_{\mu\nu} &= \tfrac{1}{2}(B_{\mu\nu} - B_{\mu\nu}) \\ \sigma_{\mu\nu} &= \tfrac{1}{2}(B_{\mu\nu} + B_{\mu\nu}) - \tfrac{1}{3}\theta h_{\mu\nu} \\ \theta &= h^{\mu\nu} B_{\mu\nu} = g^{\mu\nu} B_{\mu\nu} = \nabla_\mu u^\mu \ \ . \end{aligned} \tag{8.59}$$

The quantities $\omega_{\mu\nu}$, $\sigma_{\mu\nu}$ and θ are known as the *rotation tensor*, *shear tensor*, and *expansion* of the congruence (family) of geodesics defined by u^μ. In particular, the geodesics will converge (diverge) if $\theta > 0$ ($\theta < 0$). In terms of these quantities we can write the Raychaudhuri equation (8.47) as

$$\frac{d}{d\tau}\theta = -\tfrac{1}{3}\theta^2 - \sigma^{\mu\nu}\sigma_{\mu\nu} + \omega^{\mu\nu}\omega_{\mu\nu} - R_{\mu\nu} u^\mu u^\nu \ \ . \tag{8.60}$$

<u>Remarks:</u>

1. An important special case of this equation arises when the rotation is zero, $\omega_{\mu\nu} = 0$. This happens for example when $u_\mu = \partial_\mu F$ is the gradient co-vector of some function F. In this case u_μ is orthogonal to the level-surfaces of F.

   In this case one has

$$\frac{d}{d\tau}\theta = -\tfrac{1}{3}\theta^2 - \sigma^{\mu\nu}\sigma_{\mu\nu} - R_{\mu\nu}u^\mu u^\nu \ . \tag{8.61}$$

   The first two terms on the right hand side are manifestly non-positive (recall that $\sigma_{\mu\nu}$ is a spatial tensor and hence $\sigma_{\mu\nu}\sigma^{\mu\nu} \geq 0$). Thus, if one assumes that the geometry is such that

$$R_{\mu\nu}u^\mu u^\nu \geq 0 \tag{8.62}$$

   (by the Einstein equations to be discussed in the next section, this translates into a positivity condition on the energy-momentum tensor known as the *strong energy condition*), one finds

$$\frac{d}{d\tau}\theta = -\tfrac{1}{3}\theta^2 - \sigma^{\mu\nu}\sigma_{\mu\nu} - R_{\mu\nu}u^\mu u^\nu \leq 0 \ . \tag{8.63}$$

   This means that the divergence (convergence) of geodesics will decrease (increase) in time. The interpretation of this result is that gravity is an *attractive* force (for matter satisfying the strong energy condition) whose effect is to focus geodesics.

2. According to (8.63), $d\theta/d\tau$ is not only negative but actually bounded from above by

$$\frac{d}{d\tau}\theta \leq -\tfrac{1}{3}\theta^2 \ . \tag{8.64}$$

   Rewriting this equation as

$$\frac{d}{d\tau}\frac{1}{\theta} \geq \frac{1}{3} \ , \tag{8.65}$$

   one deduces immediately that

$$\frac{1}{\theta(\tau)} \geq \frac{1}{\theta(0)} + \frac{\tau}{3} \ . \tag{8.66}$$

   This has the rather dramatic implication that, if $\theta(0) < 0$ (i.e. the geodesics are initially converging), then $\theta(\tau) \to -\infty$ within finite proper time $\tau \leq 3/|\theta(0)|$.

3. If one thinks of the geodesics as trajectories of physical particles, this is obviously a rather catastrophic sitaution in which these particles will be infinitely squashed. In general, however, the divergence of θ only indicates that the family of geodesics develops what is known as a *caustic* where different geodesics meet. Nevertheless, the above result plays a crucial role in establishing the occurrence of true singularities in general relativity if supplemented e.g. by conditions which ensure that such "harmless" caustics cannot appear.

9 Towards the Einstein Equations

9.1 Heuristics

We expect the gravitational field equations to be non-linear second order partial differential equations for the metric. If we knew more about the weak field equations of gravity (which should thus be valid near the origin of an inertial coordinate system) we could use the Einstein equivalence principle (or the principle of general covariance) to deduce the equations for strong fields.

However, we do not know a lot about gravity beyond the Newtonian limit of weak time-independent fields and low velocities, simply because gravity is so 'weak'. Hence, we cannot find the gravitational field equations in a completely systematic way and some guesswork will be required.

Nevertheless we will see that with some very few natural assumptions we will arrive at an essentially unique set of equations. Further theoretical (and aesthetical) confirmation for these equations will then come from the fact that they turn out to be the Euler-Lagrange equations of the absolutely simplest action principle for the metric imaginable.

Recall that, way back, in section 1.1, we had briefly discussed the possibility of a scalar relativistic theorry of gravity described by an equation of the form (1.2)

$$\Delta\phi = 4\pi G\rho \quad \longrightarrow \quad \Box\phi = 4\pi G\rho \ . \tag{9.1}$$

We had noted there that one way to render this equation (tensorially) consistent is to think of both the left and the right hand side as (00)-components of some tensor, which we expressed in (1.5) as

$$\{\text{Some tensor generalising } \Delta\phi\}_{\alpha\beta} \sim 4\pi G T_{\alpha\beta} \ . \tag{9.2}$$

While this appeared to be an exotic proposal back in section 1.1, we now understand that this is exactly what is required, and we have a fairly precise idea of what this tensor on the left-hand-side should be.

Indeed, recall from our discussion of the Newtonian limit of the geodesic equation that the weak static field produced by a non-relativistic mass density ρ is

$$g_{00} = -(1+2\phi) \ , \tag{9.3}$$

With the identification

$$T_{00} = \rho \ , \tag{9.4}$$

the Newtonian field equation $\Delta\phi = 4\pi G\rho$ can now also be written as

$$\Delta g_{00} = -8\pi G T_{00} \ . \tag{9.5}$$

This suggests that the weak-field equations for a general energy-momentum tensor take the form

$$E_{\mu\nu} = 8\pi G T_{\mu\nu} \ , \tag{9.6}$$

where $E_{\mu\nu}$ is constructed from the metric and its first and second derivatives.

But by the Einstein equivalence principle, if this equation is valid for weak fields (i.e. near the origin of an inertial coordinate system) then also the equations which govern gravitational fields of arbitrary strength must be of this form, with $E_{\mu\nu}$ a tensor constructed from the metric and its first and second derivatives.

Another way of anticipating what form the field equations for gravity may take is via an analogy, a comparison of the geodesic deviation equations in Newton's theory and in General Relativity. Recall that in Newton's theory we have

$$\begin{aligned} \frac{d^2}{dt^2}\delta x^i &= -K^i_{\ j}\delta x^j \\ K^i_{\ j} &= \partial^i\partial_j\phi \ , \end{aligned} \tag{9.7}$$

whereas in General Relativity we have

$$\begin{aligned} \frac{D^2}{Dt^2}\delta x^\mu &= -K^\mu_{\ \nu}\delta x^\nu \\ K^\mu_{\ \nu} &= R^\mu_{\ \lambda\nu\rho}\dot{x}^\lambda\dot{x}^\rho \ . \end{aligned} \tag{9.8}$$

Now Newton's field equation is

$$\mathrm{Tr}\, K \equiv \Delta\phi = 4\pi G\rho \ , \tag{9.9}$$

while in General Relativity we have

$$\mathrm{Tr}\, K = R_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \ . \tag{9.10}$$

This suggests that somehow in the gravitational field equations of General Relativity, $\Delta\phi$ should be replaced by the Ricci tensor $R_{\mu\nu}$. Note that, at least roughly, the tensorial structure of this identification is compatible with the relation between ϕ and g_{00} in the Newtonian limit, the relation between ρ and the 0-0 component T_{00} of the energy momentum tensor, and the fact that for small velocities $R_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \sim R_{00}$.

We will now turn to a somewhat more precise argument along these lines which will enable us to determine $E_{\mu\nu}$.

9.2 A More Systematic Approach

Let us take stock of what we know about $E_{\mu\nu}$.

1. $E_{\mu\nu}$ is a tensor

2. $E_{\mu\nu}$ has the dimensions of a second derivative. If we assume that no new dimensionful constants enter in $E_{\mu\nu}$ then it has to be a linear combination of terms which are either second derivatives of the metric or quadratic in the first derivatives of the metric. (Later on, we will see that there is the possibility of a zero derivative term, but this requires a new dimensionful constant, the *cosmological constant* Λ. Higher derivative terms could in principle appear but would only be relevant at very high energies.)

3. $E_{\mu\nu}$ is symmetric since $T_{\mu\nu}$ is symmetric.

4. Since $T_{\mu\nu}$ is covariantly conserved, the same has to be true for $E_{\mu\nu}$,

$$\nabla_\mu T^{\mu\nu} = 0 \Rightarrow \nabla_\mu E^{\mu\nu} = 0 \ . \tag{9.11}$$

5. Finally, for a weak stationary gravitational field and non-relativistic matter we should find

$$E_{00} = -\Delta g_{00} \ . \tag{9.12}$$

Now it turns out that these conditions (1)-(5) determine $E_{\mu\nu}$ *uniquely*! First of all, (1) and (2) tell us that $E_{\mu\nu}$ has to be a linear combination

$$E_{\mu\nu} = aR_{\mu\nu} + bg_{\mu\nu}R \ , \tag{9.13}$$

where $R_{\mu\nu}$ is the Ricci tensor and R the Ricci scalar. Then condition (3) is automatically satisfied.

To implement (4), we rewrite the above as a linear combination of the Einstein tensor (7.53) and $g_{\mu\nu}R$,

$$E_{\mu\nu} = aG_{\mu\nu} + cg_{\mu\nu}R \equiv a(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R) + cg_{\mu\nu}R \tag{9.14}$$

and recall the contracted Bianchi identity (7.51,7.52),

$$\nabla^\mu G_{\mu\nu} = 0 \ . \tag{9.15}$$

It follows that (4) is satisfied iff $c\nabla_\nu R = c\partial_\nu R = 0$. We therefore have to require either $\nabla_\nu R = 0$ or $c = 0$. That the first possibility is ruled out (inconsistent) can be seen by taking the trace of (9.6),

$$E^\mu_\mu = (4c - a)R = 8\pi G T^\mu_\mu \ . \tag{9.16}$$

Thus, R is proportional to T^μ_μ and since this quantity need certainly not be constant for a general matter configuration, we are led to the conclusion that $c = 0$. Thus we find

$$E_{\mu\nu} = aG_{\mu\nu} \ . \tag{9.17}$$

We can now use condition (5) to determine the constant a.

9.3 The Weak-Field Limit

By the above considerations we have determined the field equations to be of the form

$$aG_{\mu\nu} = 8\pi G T_{\mu\nu} \quad , \tag{9.18}$$

with a some, as yet undetermined, constant. We will now consider the weak-field limit of this equation. We need to find that G_{00} is proportional to Δg_{00} and we can then use the condition (5) to fix the value of a. The following manipulations are somewhat analogous to those we performed when considering the Newtonian limit of the geodesic equation. The main difference is that now we are dealing with second derivatives of the metric rather than with just its first derivatives entering in the geodesic equation.

First of all, for a non-relativistic system we have $|T_{ij}| \ll T_{00}$ and hence $|G_{ij}| \ll |G_{00}|$. Therefore we conclude

$$|T_{ij}| \ll T_{00} \Rightarrow R_{ij} \sim \tfrac{1}{2} g_{ij} R \quad . \tag{9.19}$$

Next, for a weak field we have $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu}$ small (in a suitable sense) and, in particular,

$$R \sim \eta^{\mu\nu} R_{\mu\nu} = R^k_{\ k} - R_{00} \quad , \tag{9.20}$$

which, together with (9.19), translates into

$$R \sim \tfrac{3}{2} R - R_{00} \quad . \tag{9.21}$$

or

$$R \sim 2R_{00} \quad . \tag{9.22}$$

In the weak field limit, R_{00} in turn is given by

$$R_{00} = R^k_{\ 0k0} = \eta^{ik} R_{i0k0} \quad . \tag{9.23}$$

Moreover, in this limit only the linear (second derivative) part of $R_{\mu\nu\lambda\sigma}$ will contribute, not the terms quadratic in first derivatives. Thus we can use the expression (7.11) for the curvature tensor. Additionally, in the static case we can ignore all time derivatives. Then only one term (the third) of (7.11) contributes and we find

$$R_{i0k0} = -\tfrac{1}{2} g_{00,ik} \quad , \tag{9.24}$$

and therefore

$$R_{00} = -\tfrac{1}{2} \Delta g_{00} \quad . \tag{9.25}$$

Thus, putting everything together, we get

$$\begin{aligned} G_{00} &= (R_{00} - \tfrac{1}{2} g_{00} R) \\ &= (R_{00} - \tfrac{1}{2} \eta_{00} R) \\ &= (R_{00} + \tfrac{1}{2} R) \\ &= (R_{00} + R_{00}) \\ &= -\Delta g_{00} \quad . \end{aligned} \tag{9.26}$$

Thus we obtain the correct functional form of E_{00} and comparison with condition (5) determines $a = +1$ and therefore $E_{\mu\nu} = G_{\mu\nu}$.

9.4 The Einstein Equations

We have finally arrived at the Einstein equations for the gravitational field (metric) of a matter-energy configuration described by the energy-momentum tensor $T_{\mu\nu}$. It is

$$\boxed{R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = 8\pi G\, T_{\mu\nu}} \tag{9.27}$$

With c not set equal to one, and with the convention that T_{00} is normalised such that it gives the energy-density rather than the mass-density, one finds that the factor $8\pi G$ on the right hand side should be replaced by

$$8\pi G \quad \rightarrow \quad \frac{8\pi G}{c^4} \quad . \tag{9.28}$$

A note on dimensions: Newton's constant has dimensions (M mass, L length, T time) $[G] = \mathsf{M}^{-1}\mathsf{L}^3\mathsf{T}^{-2}$ so that

$$[G] = \mathsf{M}^{-1}\mathsf{L}^3\mathsf{T}^{-2} \quad \Rightarrow \quad [G/c^4] = \mathsf{L}^{-1}\mathsf{M}^{-1}\mathsf{T}^2 \quad . \tag{9.29}$$

Moreover, an energy density $\rho = \mu c^2$, μ a mass density, has dimensions

$$[\rho] = [\mu c^2] = \mathsf{M}\mathsf{L}^{-3}\mathsf{L}^2\mathsf{T}^{-2} = \mathsf{M}\mathsf{L}^{-1}\mathsf{T}^{-2} \quad . \tag{9.30}$$

Thus

$$[\rho G/c^4] = \mathsf{L}^{-2} = [R_{\mu\nu}] \quad , \tag{9.31}$$

as it should be. Frequently, an alternative convention is used in which T_{00} is a mass density, so that then $T_{tt} = c^2 T_{00}$ is the energy density. In that case, the factor on the right-hand-side of the Einstein equations is $8\pi G/c^2$.

Another common way of writing the Einstein equations is obtained by taking the trace of (9.27), which yields

$$R - 2R = 8\pi G\, T^{\mu}_{\ \mu} \quad , \tag{9.32}$$

and substituting this back into (9.27) to obtain

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T^{\lambda}_{\ \lambda}) \quad . \tag{9.33}$$

In particular, for the vacuum, $T_{\mu\nu} = 0$, the Einstein equations are simply

$$R_{\mu\nu} = 0 \quad . \tag{9.34}$$

A space-time metric satisfying this equation is, for obvious reasons, said to be Ricci-flat. And I should probably not have said 'simply' in the above because even the vacuum Einstein equations still constitute a complicated set of non-linear coupled partial differential equations whose general solution is not, and probably will never be, known. Usually one makes some assumptions, in particular regarding the symmetries of the metric, which simplify the equations to the extent that they can be analysed explicitly, either analytically, or at least qualitatively or numerically.

As we saw before, in two and three dimensions, vanishing of the Ricci tensor implies the vanishing of the Riemann tensor. Thus in these cases, the space-times are necessarily flat away from where there is matter, i.e. at points at which $T_{\mu\nu}(x) = 0$. Thus there are no true gravitational fields and no gravitational waves.

In four dimensions, however, the situation is completely different. As we saw, the Ricci tensor has 10 independent components whereas the Riemann tensor has 20. Thus there are 10 components of the Riemann tensor which can curve the vacuum, as e.g. in the field around the sun, and a lot of interesting physics is already contained in the vacuum Einstein equations.

9.5 Significance of the Bianchi Identities

Because the Ricci tensor is symmetric, the Einstein equations consitute a set of ten algebraically independent second order differential equations for the metric $g_{\mu\nu}$. At first, this looks exactly right as a set of equations for the ten components of the metric.

But at second sight, this cannot be right. After all, the Einstein equations are generally covariant, so that they can at best determine the metric up to coordinate transformations. Therefore we should only expect six independent generally covariant equations for the metric. Here we should recall the contracted Bianchi identities. They tell us that

$$\nabla^{\mu} G_{\mu\nu} = 0 \quad , \tag{9.35}$$

and hence, even though the ten Einstein equations are algebraically independent, there are four differential relations among them, so this is just right.

It is no coincidence, by the way, that the Bianchi identities come to the rescue of general covariance. We will see later that the Bianchi identities can in fact be understood as a consequence of the general covariance of the Einstein equations (and of the corresopnding action principle).

The general covariance of the Einstein equations is reflected in the fact that only six of the ten equations are truly dynamical equations, namely (for the vacuum equations for simplicity)

$$G_{ij} = 0 \quad , \tag{9.36}$$

where $i, j = 1, 2, 3$. The other four, namely

$$G_{\mu 0} = 0 \quad , \tag{9.37}$$

are constraints that have to be satisfied by the initial data g_{ij} and, say, dg_{ij}/dt on some initial spacelike hypersurface (Cauchy surface).

These constraints are analogues of the Gauss law constraint of Maxwell theory (which is a consequence of the $U(1)$ gauge invariance of the theory), but significantly more complicated. Over the years, a lot of effort has gone into developing a formalism and framework for the initial value and canonical (phase space) description of General Relativity. The most well known and useful of these is the so-called ADM (Arnowitt, Deser, Misner) formalism. The canonical formalism has been developed in particular with an eye towards canonical quantisation of gravity.

9.6 The Cosmological Constant

As mentioned before, there is one more term that can be added to the Einstein equations provided that one relaxes the condition (2) that only terms quadratic in derivatives should appear. This term takes the form $\Lambda g_{\mu\nu}$. This is compatible with the condition (4) (the conservation law) provided that Λ is a constant, the *cosmological constant.* it is a dimensionful parameter with dimension $[\Lambda] = \mathsf{L}^{-2}$ one over length squared.

The Einstein equations with a cosmological constant now read

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \quad . \tag{9.38}$$

To be compatible with condition (5) ((1), (3) and (4) are obviously satisfied), Λ has to be quite small (and observationally it is very small indeed).

Λ plays the role of a vacuum energy density, as can be seen by writing the vacuum Einstein equations as

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -\Lambda g_{\mu\nu} \quad . \tag{9.39}$$

Comparing this with the energy-momentum tensor of, say, a perfect fluid (see the section on Cosmology),

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} \quad , \tag{9.40}$$

we see that Λ corresponds to the energy-density and pressure values

$$\rho_\Lambda = -p_\Lambda = \frac{\Lambda}{8\pi G} \quad . \tag{9.41}$$

The cosmological constant was originally introduced by Einstein because he was unable to find static cosmological solutions without it. After Hubble's discovery of the expansion of the universe, a static universe fell out of fashion, the cosmological constant was no longer required and Einstein rejected it (supposedly calling the introduction of Λ in the first place his biggest blunder because he could have *predicted* the expansion of the universe if he had simply believed in his equations without the cosmological constant).

However, things are not as simple as that. In fact, one of the biggest puzzles in theoretical physics today is why the cosmological constant is so small. According to standard quantum field theory lore, the vacuum energy density should be many many orders of magnitude larger than astrophysical observations allow. Now usually in quantum field theory one does not worry too much about the vacuum energy as one can normal-order it away. However, as we know, gravity is unlike any other theory in that not only energy-differences but absolute energies matter (and cannot just be dropped). The question why the observed cosmological constant is so small (it may be exactly zero, but recent astrophysical observations appear to favour a tiny non-zero value) is known as the *Cosmological Constant Problem.* We will consider the possibility that $\Lambda \neq 0$ only in the section on Cosmology (in all other applications, Λ can indeed be neglected).

9.7 * The Weyl Tensor and the Propagation of Gravity

The Einstein equations

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{9.42}$$

can, taken at face value, be regarded as ten algebraic equations for certain traces of the Riemann tensor $R_{\mu\nu\rho\sigma}$. But $R_{\mu\nu\rho\sigma}$ has, as we know, twenty independent components, so how are the other ten determined? The obvious answer, already given above, is of course that we solve the Einstein equations for the metric $g_{\mu\nu}$ and then calculate the Riemann curvature tensor of that metric.

However, this answer leaves something to be desired because it does not really provide an explanation of how the information about these other components is encoded in the Einstein equations. It is interesting to understand this because it is precisely these components of the Riemann tensor wich represent the effects of gravity in vacuum, i.e. where $T_{\mu\nu} = 0$, like tidal forces and gravitational waves.

The more insightful answer is that the information is encoded in the Bianchi identities which serve as propagation equations for the trace-free parts of the Riemann tensor away from the regions where $T_{\mu\nu} \neq 0$.

Let us see how this works. First of all, we need to decompose the Riemann tensor into its trace parts $R_{\mu\nu}$ and R (determined directly by the Einstein equations) and its traceless part $C_{\mu\nu\rho\sigma}$, the *Weyl tensor*.

In any $n \geq 4$ dimensions, the Weyl tensor is defined by

$$\begin{aligned} C_{\mu\nu\rho\sigma} &= R_{\mu\nu\rho\sigma} \\ &- \frac{1}{n-2}(g_{\mu\rho}R_{\nu\sigma} + R_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}R_{\mu\sigma} - R_{\nu\rho}g_{\mu\sigma}) \\ &+ \frac{1}{(n-1)(n-2)}R(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}) \ . \end{aligned} \tag{9.43}$$

This definition is such that $C_{\mu\nu\rho\sigma}$ has all the symmetries of the Riemann tensor (this is manifest) and that all of its traces are zero, i.e.

$$C^{\mu}_{\ \nu\mu\sigma} = 0 \ . \tag{9.44}$$

In the vacuum, $R_{\mu\nu} = 0$, and therefore

$$T_{\mu\nu}(x) = 0 \ \Rightarrow \ R_{\mu\nu\rho\sigma}(x) = C_{\mu\nu\rho\sigma}(x) \ , \tag{9.45}$$

and, as anticipated, the Weyl tensor encodes the information about the gravitational field in vacuum. The question thus is how $C_{\mu\nu\rho\sigma}$ is determined everywhere in space-time by an energy-momentum tensor which may be localised in some finite region of space-time.

Contracting the Bianchi identity, which we write as

$$\nabla_{[\lambda}R_{\mu\nu]\rho\sigma} = 0 \ , \tag{9.46}$$

over λ and ρ and making use of the symmetries of the Riemann tensor, one obtains

$$\nabla^{\lambda}R_{\lambda\sigma\mu\nu} = \nabla_{\mu}R_{\nu\sigma} - \nabla_{\nu}R_{\mu\sigma} \ . \tag{9.47}$$

Expressing the Riemann tensor in terms of its contractions and the Weyl tensor, and using the Einstein equations to replace the Ricci tensor and Ricci scalar by the energy-momentum tensor, one now obtains a propagation equation for the Weyl tensor of the form

$$\nabla^{\mu}C_{\mu\nu\rho\sigma} = J_{\nu\rho\sigma} \ , \tag{9.48}$$

where $J_{\nu\rho\sigma}$ depends only on the energy-momentum tensor and its derivatives. Determining $J_{\nu\rho\sigma}$ in this way is straightforward and one finds

$$J_{\nu\rho\sigma} = \kappa\frac{n-3}{n-2}\left[\nabla_{\rho}T_{\nu\sigma} - \nabla_{\sigma}T_{\nu\rho} - \frac{1}{n-1}\left[\nabla_{\rho}T^{\lambda}_{\lambda}g_{\nu\sigma} - \nabla_{\sigma}T^{\lambda}_{\lambda}g_{\nu\rho}\right]\right] \ . \tag{9.49}$$

The equation (9.48) is reminiscent of the Maxwell equation

$$\nabla^{\mu}F_{\mu\nu} = J_{\nu} \ , \tag{9.50}$$

and this is the starting point for a very fruitful analogy between the two subjects. Indeed it turns out that in many other respects as well $C_{\mu\nu\rho\sigma}$ behaves very much like an electro-magnetic field: one can define electric and magnetic components E and B, these satisfy $|E| = |B|$ for a gravitational wave, etc.

Finally, the Weyl tensor is also useful in other contexts as it is *conformally invariant*, i.e. $C^{\mu}_{\nu\rho\sigma}$ is invariant under conformal rescalings of the metric

$$g_{\mu\nu}(x) \to \mathrm{e}^{f(x)} g_{\mu\nu}(x) \quad . \tag{9.51}$$

In particular, the Weyl tensor is zero if the metric is *conformally flat*, i.e. related by a conformal transformation to the flat metric, and conversely vanishing of the Weyl tensor is also a sufficient condition for a metric to be conformal to the flat metric.

10 The Einstein Equations from a Variational Principle

10.1 The Einstein-Hilbert Action

To increase our confidence that the Einstein equations we have derived above are in fact reasonable and almost certainly correct, we can adopt a more modern point of view. We can ask if the Einstein equations follow from an action principle or, alternatively, what would be a natural action principle for the metric.

After all, for example in the construction of the Standard Model, one also does not start with the equations of motion but one writes down the simplest possible Lagrangian with the desired field content and symmetries.

We will start with the gravitational part, i.e. the Einstein tensor $G_{\mu\nu}$ of the Einstein equations, and deal with the matter part, the energy-momentum tensor $T_{\mu\nu}$, later.

By general covariance, an action for the metric $g_{\mu\nu}$ will have to take the form

$$S = \int \sqrt{g} d^4x \, \Phi(g_{\mu\nu}) \quad , \tag{10.1}$$

where Φ is a scalar constructed from the metric. So what is Φ going to be? Clearly, the simplest choice is the Ricci scalar R, and this is also the unique choice if one is looking for a scalar constructed from not higher than second derivatives of the metric. Therefore we postulate the beautifully simple and elegant action

$$\boxed{S_{EH} = \int \sqrt{g} d^4x \, R} \tag{10.2}$$

known as the Einstein-Hilbert action. It was presented by Hilbert practically on the same day that Einstein presented his final form (9.27) of the gravitational field equations. Discussions regarding who did what first and who deserves credit for what have been a favourite occupation of historians of science ever since. But Hilbert's work would certainly not have been possible without Einstein's realisation that gravity should be regarded not as a force but as a property of space-time and that Riemannian geometry provides the correct framework for embodying the equivalence principle.

We will now prove that the Euler-Lagrange equations following from the Einstein-Hilbert Lagrangian indeed give rise to the Einstein tensor and the vacuum Einstein equations. It is truly remarkable, that such a simple Lagrangian is capable of explaining practically all known gravitational, astrophysical and cosmological phenomena (contrast this with the complexity of the Lagrangian of the Standard Model or any of its generalisations).

Since the Ricci scalar is $R = g^{\mu\nu} R_{\mu\nu}$, it is simpler to consider variations $\delta g^{\mu\nu}$ of the inverse metric instead of $\delta g_{\mu\nu}$. Thus, as a first step we write

$$\begin{aligned} \delta S_{EH} &= \delta \int \sqrt{g} d^4x \; g^{\mu\nu} R_{\mu\nu} \\ &= \int d^4x \; (\delta\sqrt{g} g^{\mu\nu} R_{\mu\nu} + \sqrt{g}\delta g^{\mu\nu} R_{\mu\nu} + \sqrt{g} g^{\mu\nu} \delta R_{\mu\nu}) \; . \end{aligned} \tag{10.3}$$

Now we make use of the identity (exercise!)

$$\delta g^{1/2} = \tfrac{1}{2} g^{1/2} g^{\lambda\rho} \delta g_{\lambda\rho} = -\tfrac{1}{2} g^{1/2} g_{\lambda\rho} \delta g^{\lambda\rho} \; . \tag{10.4}$$

Hence,

$$\begin{aligned} \delta S_{EH} &= \int \sqrt{g} d^4x \; [(-\tfrac{1}{2} g_{\mu\nu} R + R_{\mu\nu}) \delta g^{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu}] \\ &= \int \sqrt{g} d^4x \; (R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R) \delta g^{\mu\nu} + \int \sqrt{g} d^4x \; g^{\mu\nu} \delta R_{\mu\nu} \; . \end{aligned} \tag{10.5}$$

The first term all by itself would already give the Einstein tensor. Thus we need to show that the second term is identically zero. I do not know of any particularly elegant argument to establish this (in a coordinate basis - written in terms of differential forms this would be completely obvious), so this will require a little bit of work, but it is not difficult.

Postponing the proof of this statement to the end of this section, we have established that the variation of the Einstein-Hilbert action gives the gravitational part (left hand side) of the Einstein equations,

$$\delta \int \sqrt{g} d^4x \; R = \int \sqrt{g} d^4x \; (R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R) \delta g^{\mu\nu} \; . \tag{10.6}$$

<u>REMARKS:</u>

1. We can also write this as

$$\frac{1}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} S_{EH} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R \; . \tag{10.7}$$

2. If one wants to include the cosmological constant Λ, then the action gets modified to

$$S_{EH,\Lambda} = \int \sqrt{g} d^4x \; (R - 2\Lambda) \; . \tag{10.8}$$

   Indeed, the only effect of including Λ is to replace $R \to R - 2\Lambda$ in the Einstein equations, so that

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R \to G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} (R - 2\Lambda) = G_{\mu\nu} + \Lambda g_{\mu\nu} \; , \tag{10.9}$$

   which gives rise to the modified Einstein equation (9.38).

3. Of course, once one is working at the level of the action, it is easy to come up with covariant generalisations of the Einstein-Hilbert action, such as

$$S = \int \sqrt{g} d^4x \; (R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R \Box R + \ldots) \;\; , \tag{10.10}$$

but these invariably involve higher-derivative terms and are therefore irrelevant for low-energy physics and thus the world we live in. Such terms could be relevant for the early universe, however, and are also typically predicted by quantum theories of gravity like string theory.

We conclude this section with the proof that $g^{\mu\nu}\delta R_{\mu\nu}$ is a total derivative.

First of all, we need the explicit expression for the Ricci tensor in terms of the Christoffel symbols, which can be obtained by contraction of (7.5),

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\ \mu\nu} - \partial_\nu \Gamma^\lambda_{\ \mu\lambda} + \Gamma^\lambda_{\ \lambda\rho} \Gamma^\rho_{\ \nu\mu} - \Gamma^\lambda_{\ \nu\rho} \Gamma^\rho_{\ \lambda\mu} \;\; . \tag{10.11}$$

Now we need to calculate the variation of $R_{\mu\nu}$. We will not require the explicit expression in terms of the variations of the metric, but only in terms of the variations $\delta\Gamma^\mu_{\ \nu\lambda}$ induced by the variations of the metric. This simplifies things considerably.

Obviously, $\delta R_{\mu\nu}$ will then be a sum of six terms,

$$\delta R_{\mu\nu} = \partial_\lambda \delta\Gamma^\lambda_{\ \mu\nu} - \partial_\nu \delta\Gamma^\lambda_{\ \mu\lambda} + \delta\Gamma^\lambda_{\ \lambda\rho} \Gamma^\rho_{\ \nu\mu} + \Gamma^\lambda_{\ \lambda\rho} \delta\Gamma^\rho_{\ \nu\mu} - \delta\Gamma^\lambda_{\ \nu\rho} \Gamma^\rho_{\ \lambda\mu} - \Gamma^\lambda_{\ \nu\rho} \delta\Gamma^\rho_{\ \lambda\mu} \;\; . \tag{10.12}$$

Now the crucial observation is that $\delta\Gamma^\mu_{\ \nu\lambda}$ is a tensor. This follows from the arguments given at the end of section 4, under the heading 'Generalisations', but I will repeat it here in the present context. Of course, we know that the Christoffel symbols themselves are not tensors, because of the inhomogeneous (second derivative) term appearing in the transformation rule under coordinate transformations. But this term is independent of the metric. Thus the metric variation of the Christoffel symbols indeed transforms as a tensor.

This can also be confirmed by explicit calculation. Just for the record, I will give an expression for $\delta\Gamma^\mu_{\ \nu\lambda}$ which is easy to remember as it takes exactly the same form as the definition of the Christoffel symbol, only with the metric replaced by the metric variation and the partial derivatives by covariant derivatives, i.e.

$$\delta\Gamma^\mu_{\ \nu\lambda} = \tfrac{1}{2} g^{\mu\rho} (\nabla_\nu \delta g_{\rho\lambda} + \nabla_\lambda \delta g_{\rho\nu} - \nabla_\rho \delta g_{\nu\lambda}) \;\; . \tag{10.13}$$

It turns out, none too surprisingly, that $\delta R_{\mu\nu}$ can be written rather compactly in terms of covariant derivatives of $\delta\Gamma^\mu_{\ \nu\lambda}$, namely as

$$\delta R_{\mu\nu} = \nabla_\lambda \delta\Gamma^\lambda_{\ \mu\nu} - \nabla_\nu \delta\Gamma^\lambda_{\ \lambda\mu} \;\; . \tag{10.14}$$

As a first check on this, note that the first term on the right hand side is manifestly symmetric and that the second term is also symmetric because of (4.36) and (5.45). To establish (10.14), one simply has to use the definition of the covariant derivative. The first term is

$$\nabla_\lambda \delta\Gamma^\lambda_{\ \mu\nu} = \partial_\lambda \delta\Gamma^\lambda_{\ \mu\nu} + \Gamma^\lambda_{\ \lambda\rho}\delta\Gamma^\rho_{\ \mu\nu} - \Gamma^\rho_{\ \mu\lambda}\delta\Gamma^\lambda_{\ \rho\nu} - \Gamma^\rho_{\ \nu\lambda}\delta\Gamma^\lambda_{\ \rho\mu} \ , \tag{10.15}$$

which takes care of the first, fourth, fifth and sixth terms of (10.12). The remaining terms are

$$-\partial_\nu \delta\Gamma^\lambda_{\ \mu\lambda} + \delta\Gamma^\lambda_{\ \lambda\rho}\Gamma^\rho_{\ \nu\mu} = -\nabla_\nu \delta\Gamma^\lambda_{\ \mu\lambda} \ , \tag{10.16}$$

which establishes (10.14). Now what we really need is $g^{\mu\nu}\delta R_{\mu\nu}$,

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\lambda (g^{\mu\nu}\delta\Gamma^\lambda_{\ \mu\nu}) - \nabla_\nu (g^{\mu\nu}\delta\Gamma^\lambda_{\ \lambda\mu}) \ . \tag{10.17}$$

Using the explicit expression for $\delta\Gamma^\mu_{\ \nu\lambda}$ given above, we see that we can also write this rather neatly and compactly as

$$g^{\mu\nu}\delta R_{\mu\nu} = (\nabla^\mu\nabla^\nu - \Box g^{\mu\nu})\delta g_{\mu\nu} \ . \tag{10.18}$$

Since both of these terms are covariant divergences of vector fields,

$$\begin{aligned} g^{\mu\nu}\delta R_{\mu\nu} &= \nabla_\lambda J^\lambda \\ J^\lambda &= g^{\mu\nu}\delta\Gamma^\lambda_{\ \mu\nu} - g^{\mu\lambda}\delta\Gamma^\nu_{\ \nu\mu} \ , \end{aligned} \tag{10.19}$$

we can use Gauss' theorem to conclude that

$$\int \sqrt{g} d^4x \ g^{\mu\nu}\delta R_{\mu\nu} = \int \sqrt{g} d^4x \ \nabla_\lambda J^\lambda = 0 \ , \tag{10.20}$$

(since J^μ is constructed from the variations of the metric which, by assumption, vanish at infinity) as we wanted to show.

10.2 The Matter Action

In order to obtain the non-vacuum Einstein equations, we need to decide what the matter Lagrangian should be. Now there is an obvious choice for this. If we have matter, then in addition to the Einstein equations we also want the equations of motion for the matter fields. Therefore we should add to the Einstein-Hilbert action the standard matter action $S_M[\phi, g_{\alpha\beta}]$ (ϕ representing any kind of (vector, tensor, ...) field), of course suitably covariantised via the principle of minimal coupling (section 5). Thus the matter action for a Klein-Gordon field would be (5.4) and that for Maxwell theory would be (5.23) etc.

Of course, the variation of the matter action with respect to the matter fields will give rise to the covariant equations of motion of the matter fields. But if we want to add the matter action to the Einstein-Hilbert action and treat the metric as an additional dynamical variable, then we have to ask what the variation of the matter action with respect to the metric is. The short answer is: the energy-momentum tensor. Indeed, even though there are other definitions of the energy-momentum tensor you may know (defined via Noether's theorem applied to translations in flat space, for example), this is the modern, and by far the most useful, definition of the energy-momentum tensor, namely as the response of the matter action to a variation of the metric. And we had already seen in section 5 that this definition (with a judicious factor of 2),

$$\delta_{\text{metric}} S_M[\phi, g_{\alpha\beta}] = -\tfrac{1}{2} \int \sqrt{g} d^4x\, T_{\mu\nu} \delta g^{\mu\nu} \quad , \tag{10.21}$$

or

$$T_{\mu\nu} := -\frac{2}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} S_M[\phi, g_{\alpha\beta}] \tag{10.22}$$

reproduces the known results in the case of a scalar field or Maxwell theory. One of the many advantages of this definition is that it automatically gives a symmetric and gauge invariant tensor (no improvement terms or similar gymnastics required) which is also automatically covariantly conserved. We will establish this latter fact below - it is simply a consequence of the general covariance of S_M.

Therefore, the complete gravity-matter action for General Relativity is

$$S[g_{\alpha\beta}, \phi] = \frac{1}{16\pi G} S_{EH}[g_{\alpha\beta}] + S_M[g_{\alpha\beta}, \phi] \tag{10.23}$$

with

$$\frac{\delta S[g_{\alpha\beta}, \phi]}{\delta g^{\mu\nu}} = 0 \quad \Leftrightarrow \quad G_{\alpha\beta} = 8\pi G T_{\alpha\beta} \quad . \tag{10.24}$$

<u>REMARKS:</u>

1. As we saw above, a cosmological constant term can be included by adding a constant term to the Einstein-Hilbert Lagrangian. But one can equally well add a constant term to the matter Lagrangian instead (and this clearly reveals its interpretation as a vacuum energy of the matter fields).

2. If one were to try to deduce the gravitational field equations by starting from a variational principle, i.e. by constructing the simplest generally covariant action for the metric and the matter fields (and this would be the modern approach to the problem, had Einstein not already solved it for us a 100 years ago), then one would also invariably be led to the above action.

   The relative numerical factor $16\pi G$ between the two terms would of course then not be fixed a priori, because this appraoch will not determine Newton's constant. The prefactor could once again be determined by looking at the Newtonian limit of the resulting equations of motion.

3. Typically, the above action principle will lead to a very complicated coupled system of equations for the metric and the matter fields because the metric also appears in the energy-momentum tensor and in the equations of motion for the matter fields.

I mentioned before that it is no accident that the Bianchi identities come to the rescue of the general covariance of the Einstein equations in the sense that they reduce the number of independent equations from ten to six. We will now see that indeed the Bianchi identities are a consequence of the general covariance of the Einstein-Hilbert action. Virtually the same calculation will show that the energy-momentum tensor, as defined above, is automatically conserved (on shell) by virtue of the general covariance of the matter action.

Let us start with the Einstein-Hilbert action. We already know that

$$\delta S_{EH} = \int \sqrt{g} d^4x \; G_{\mu\nu} \delta g^{\mu\nu} \tag{10.25}$$

for any metric variation. We also know that the Einstein-Hilbert action is invariant under coordinate transformations. In particular, therefore, the above variation should be identically zero for variations of the metric induced by an infinitesimal coordinate transformation. But we know from the discussion of the Lie derivative that such a variation is of the form

$$\delta_V g_{\mu\nu} = L_V g_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu \;\; , \tag{10.26}$$

or

$$\delta_V g^{\mu\nu} = L_V g^{\mu\nu} = -(\nabla^\mu V^\nu + \nabla^\nu V^\mu) \;\; , \tag{10.27}$$

where the vector field V is the infinitesimal generator of the coordinate transformation. Thus, $\delta_V S_{EH}$ should be identically zero. Calculating this we find

$$\begin{aligned} 0 &= \delta_V S_{EH} \\ &= -\int \sqrt{g} d^4x \; G_{\mu\nu} (\nabla^\mu V^\nu + \nabla^\nu V^\mu) \\ &= -2 \int \sqrt{g} d^4x \; G_{\mu\nu} \nabla^\mu V^\nu \\ &= 2 \int \sqrt{g} d^4x \; \nabla^\mu G_{\mu\nu} V^\nu \;\; . \end{aligned} \tag{10.28}$$

Since this has to hold for all V we deduce

$$\delta_V S_{EH} = 0 \quad \forall\, V \quad \Rightarrow \quad \nabla^\mu G_{\mu\nu} = 0 \;\; , \tag{10.29}$$

and, as promised, the Bianchi identities are a consequence of the general covariance of the Einstein-Hilbert action.

Now let us play the same game with the matter action S_M. Let us denote the matter fields generically by Φ so that $L_M = L_M(\Phi, g_{\mu\nu})$. Once again, the variation $\delta_V S_M$, expressed in terms of the Lie derivatives $L_V g_{\mu\nu}$ and $\delta_V \Phi = L_V \Phi$ of the matter fields should be identically zero, by general covariance of the matter action. Proceeding as before, we find

$$\begin{aligned} 0 &= \delta_V S_M \\ &= \int \sqrt{g} d^4x \, (-T_{\mu\nu}\delta_V g^{\mu\nu} + \frac{\delta L_M}{\delta\Phi}\delta_V\Phi) \\ &= -2\int \sqrt{g}d^4x \, (\nabla^\mu T_{\mu\nu})V^\nu + \int \sqrt{g}d^4x \, \frac{\delta L_M}{\delta\Phi}\delta_V\Phi \quad . \end{aligned} \tag{10.30}$$

Now once again this has to hold for all V, and as the second term is identically zero 'on-shell', i.e. for Φ satisfying the matter equations of motion, we deduce that

$$\delta_V S_M = 0 \quad \forall\, V \quad \Rightarrow \quad \nabla^\mu T_{\mu\nu} = 0 \quad \text{on-shell} \quad . \tag{10.31}$$

This should be contrasted with the Bianchi identities which are valid 'off-shell'.

Part II: Selected Applications of General Relativity

Until now, our treatment of the basic structures and properties of Riemannian geometry and General Relativity has been rather systematic. In the second half of the course, we will instead discuss some selected applications of General Relativity. These will include, of course, a discussion of the classical predictions and tests of General Relativity (the deflection of light by the sun and the perihelion shift of Mercury). Then one can, depending on time and interest, look at a variety of other topics. Here is a list of the topics covered in the remainder of the notes (the blocks made of sections 11-13, 14-18, 19-20, and 21-22 can be read independently of each other):

11. The Schwarzschild Metric
12. Particle and Photon Orbits in the Schwarzschild Geometry
13. Black Holes: Approaching and Crossing the Schwarzschild Radius
14. Interlude: Maximally Symmetric Spaces
15. Cosmology I: Basics
16. Cosmology II: Basics of Friedmann-Robertson-Walker Cosmology
17. Cosmology III: Qualitative Analysis
18. Cosmology IV: Exact Solutions
19. Linearised Gravity and Gravitational Waves
20. Exact Wave-like Solutions of the Einstein Equations
21. Kaluza-Klein Theory

11 The Schwarzschild Metric

11.1 Introduction

Einstein himself suggested three tests of General Relativity, namely

1. the gravitational red-shift
2. the deflection of light by the sun
3. the anomalous precession of the perihelion of the orbits of Mercury and Venus,

and calculated the theoretical predictions for these effects. In the meantime, other tests have also been suggested and performed, for example the time delay of radar echos passing the sun (the Shapiro effect).

All these tests have in common that they are carried out in empty space, with gravitational fields that are to an excellent aproximation static (time independent) and isotropic (spherically symmetric). Thus our first aim will have to be to solve the vacuum Einstein equations under the simplifying assumptions of isotropy and time-independence. This, as we will see, is indeed not too difficult.

11.2 Static Isotropic Metrics

Even though we have decided that we are interested in static isotropic metrics, we still have to determine what we actually mean by this statement. After all, a metric which looks time-independent in one coordinate system may not do so in another coordinate system. There are two ways of approaching this issue:

1. One can try to look for a covariant characterisation of such metrics, in terms of Killing vectors etc. In the present context, this would amount to considering metrics which admit four Killing vectors, one of which is timelike, with the remaining three representing the Lie algebra of the rotation group $SO(3)$.
2. Or one works with 'preferred' coordinates from the outset, in which these symmetries are manifest.

While the former approach may be conceptually more satisfactory, the latter is much easier to work with and is hence the one we will adopt. We will implement the condition of time-independence by choosing all the components of the metric to be time-independent, and we will express the condition of isotropy by the requirement that, in terms of spatial polar coordinates (r, θ, ϕ) the metric can be written as

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + 2C(r)dr\,dt + D(r)r^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{11.1}$$

This ansatz, depending on the four functions $A(r), B(r), C(r), D(r)$, can still be simplified a lot by choosing appropriate new time and radial coordinates.

First of all, let us introduce a new time coordinate $T(t,r)$ by

$$T(t,r) = t + \psi(r) \quad . \tag{11.2}$$

Then

$$dT^2 = dt^2 + \psi'^2 dr^2 + 2\psi' dr\, dt \quad . \tag{11.3}$$

Thus we can eliminate the off-diagonal term in the metric by choosing ψ to satisfy the differential equation

$$\frac{d\psi(r)}{dr} = -\frac{C(r)}{A(r)} \quad . \tag{11.4}$$

We can also eliminate $D(r)$ by introducing a new radial coordinate $R(r)$ by $R^2 = D(r)r^2$. Thus we can assume that the line element of a static isotropic metric is of the form

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad . \tag{11.5}$$

This is known as the *standard form* of a static isotropic metric. Another useful presentation, related to the above by a coordinate transformation, is

$$ds^2 = -E(r)dt^2 + F(r)(dr^2 + r^2 d\Omega^2) \quad . \tag{11.6}$$

This is the static isotropic metric in *isotropic form.* We will mostly be using the metric in the standard form (11.5). Let us note some immediate properties of this metric:

1. By our ansatz, the components of the metric are time-independent. Because we have been able to eliminate the $dtdr$-term, the metric is also invariant under time-reversal $t \to -t$.

2. The surfaces of constant t and r have the metric

$$ds^2|_{r=const.,t=const.} = r^2 d\Omega^2 \quad , \tag{11.7}$$

   and hence have the geometry of two-spheres.

3. Because $B(r) \neq 1$, we cannot identify r with the proper radial distance. However, r has the clear geometrical meaning that the two-sphere of constant r has the area $A(S_r^2) = 4\pi r^2$.

4. Also, even though the coordinate time t is not directly measurable, it can be invariantly characterised by the fact that $\partial/\partial t$ is a timelike Killing vector.

5. The functions A and B are now to be found by solving the Einstein field equations.

6. If we want the solution to be asymptotically flat (i.e. that it approaches Minkowski space for $r \to \infty$), we need to impose the boundary conditions

$$\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1 \ . \tag{11.8}$$

We will come back to other aspects of measurements of space and time in such a geometry after we have solved the Einstein equations.

We have assumed from the outset that the metric is static. However, it can be shown with little effort (see section 11.4)) that the vacuum Einstein equations imply that a spherically symmetric metric is static. This result is known as *Birkhoff's theorem.* It is the General Relativity analogue of the Newtonian result that a spherically symmetric body behaves as if all the mass were concentrated in its center. In the present context it means that the gravitational field not only of a static spherically symmetric body is static and spherically symmetric (as we have assumed), but that the same is true for a radially oscillating/pulsating object. This is a bit surprising because one would expect such a body to emit gravitational radiation. What Birkhoff's theorem shows is that this radiation cannot escape into empty space (because otherwise it would destroy the time-independence of the metric). Translated into the language of waves, this means that there is no s-wave (monopole) gravitational radiation.

11.3 Solving the Einstein Equations for a Static Spherically Symmetric Metric

We will now solve the vacuum Einstein equations for the static isotropic metric in standard form, i.e. we look for solutions of $R_{\mu\nu} = 0$. You should have already (as an exercise) calculated all the Christoffel symbols of this metric, using the Euler-Lagrange equations for the geodesic equation.

As a reminder, here is how this method works. To calculate all the Christoffel symbols $\Gamma^r_{\mu\nu}$, say, in one go, you look at the Euler Lagrange equation for $r = r(\tau)$ resulting from the Lagrangian $\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu/2$. This is easily seen to be

$$\ddot{r} + \frac{B'}{2B}\dot{r}^2 + \frac{A'}{2B}\dot{t}^2 + \cdots = 0 \tag{11.9}$$

(a prime denotes an r-derivative), from which one reads off that $\Gamma^r_{rr} = B'/2B$ etc. Proceeding in this way, you should find (or have found) that the non-zero Christoffel symbols are given by

$$\begin{aligned}
&\Gamma^r{}_{rr} = \frac{B'}{2B} \qquad \Gamma^r{}_{tt} = \frac{A'}{2B} \\
&\Gamma^r{}_{\theta\theta} = -\frac{r}{B} \qquad \Gamma^r{}_{\phi\phi} = -\frac{r\sin^2\theta}{B} \\
&\Gamma^\theta{}_{\theta r} = \Gamma^\phi{}_{\phi r} = \frac{1}{r} \qquad \Gamma^t{}_{tr} = \frac{A'}{2A} \\
&\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^\phi{}_{\phi\theta} = \cot\theta
\end{aligned} \tag{11.10}$$

Now we need to calculate the Ricci tensor of this metric. A silly way of doing this would be to blindly calculate all the components of the Riemann tensor and to then perform all the relevant contractions to obtain the Ricci tensor. A more intelligent and less time-consuming strategy is the following:

1. Instead of using the explicit formula for the Riemann tensor in terms of Christoffel symbols, one should use directly its contracted version

$$\begin{aligned} R_{\mu\nu} &= R^{\lambda}_{\ \mu\lambda\nu} \\ &= \partial_\lambda \Gamma^{\lambda}_{\ \mu\nu} - \partial_\nu \Gamma^{\lambda}_{\ \mu\lambda} + \Gamma^{\lambda}_{\ \lambda\rho}\Gamma^{\rho}_{\ \mu\nu} - \Gamma^{\lambda}_{\ \nu\rho}\Gamma^{\rho}_{\ \mu\lambda} \end{aligned} \quad (11.11)$$

   and use the formula (4.36) for $\Gamma^{\lambda}_{\ \mu\lambda}$ derived previously.

2. The high degree of symmetry of the Schwarzschild metric implies that many components of the Ricci tensor are automatically zero. For example, invariance of the Schwarzschild metric under $t \to -t$ implies that $R_{rt} = 0$. The argument for this is simple. Since the metric is invariant under $t \to -t$, the Ricci tensor should also be invariant. But under the coordinate transformation $t \to -t$, R_{rt} transforms as $R_{rt} \to -R_{rt}$. Hence, invariance requires $R_{rt} = 0$, and no further calculations for this component of the Ricci tensor are required.

3. Analogous arguments, now involving θ or ϕ instead of t, imply that

$$R_{r\theta} = R_{r\phi} = R_{t\theta} = R_{t\phi} = R_{\theta\phi} = 0 \ . \quad (11.12)$$

4. Since the Schwarzschild metric is spherically symmetric, its Ricci tensor is also spherically symmetric. It is easy to prove, by considering the effect of a coordinate transformation that is a rotation of the two-sphere defined by θ and ϕ (leaving the metric invariant), that this implies that

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta} \ . \quad (11.13)$$

   One possible proof (there may be a shorter argument): Consider a coordinate transformation $(\theta, \phi) \to (\theta', \phi')$. Then

$$d\theta^2 + \sin^2\theta d\phi^2 = \left[\left(\frac{\partial\theta}{\partial\theta'}\right)^2 + \sin^2\theta\left(\frac{\partial\phi}{\partial\theta'}\right)^2\right] d\theta'^2 + \ldots \quad (11.14)$$

   Thus, a necessary condition for the metric to be invariant is

$$(\frac{\partial\theta}{\partial\theta'})^2 + \sin^2\theta(\frac{\partial\phi}{\partial\theta'})^2 = 1 \ . \quad (11.15)$$

   Now consider the transformation behaviour of $R_{\theta\theta}$ under such a transformation. Using $R_{\theta\phi} = 0$, one has

$$R_{\theta'\theta'} = (\frac{\partial\theta}{\partial\theta'})^2 R_{\theta\theta} + (\frac{\partial\phi}{\partial\theta'})^2 R_{\phi\phi} \ . \quad (11.16)$$

Demanding that this be equal to $R_{\theta\theta}$ (because we are considering a coordinate transformation which does not change the metric) and using the condition derived above, one obtains

$$R_{\theta\theta} = R_{\theta\theta}(1 - \sin^2\theta(\frac{\partial\phi}{\partial\theta'})^2) + (\frac{\partial\phi}{\partial\theta'})^2 R_{\phi\phi} \ , \tag{11.17}$$

which implies (11.13).

5. Thus the only components of the Ricci tensor that we need to compute are R_{rr}, R_{tt} and $R_{\theta\theta}$.

Now some unenlightning calculations lead to the result that these components of the Ricci tensor are given by

$$\begin{aligned} R_{tt} &= \frac{A''}{2B} - \frac{A'}{4B}(\frac{A'}{A} + \frac{B'}{B}) + \frac{A'}{rB} \\ R_{rr} &= -\frac{A''}{2A} + \frac{A'}{4A}(\frac{A'}{A} + \frac{B'}{B}) + \frac{B'}{rB} \\ R_{\theta\theta} &= 1 - \frac{1}{B} - \frac{r}{2B}(\frac{A'}{A} - \frac{B'}{B}) \ . \end{aligned} \tag{11.18}$$

Inspection of these formulae reveals that there is a linear combination which is particularly simple, namely $BR_{tt} + AR_{rr}$, which can be written as

$$BR_{tt} + AR_{rr} = \tfrac{1}{rB}(A'B + B'A) \ . \tag{11.19}$$

Demanding that this be equal to zero, one obtains

$$A'B + B'A = 0 \ \Rightarrow \ A(r)B(r) = const. \tag{11.20}$$

Asymptotic flatness fixes this constant to be $= 1$, so that

$$B(r) = \frac{1}{A(r)} \ . \tag{11.21}$$

Plugging this result into the expression for $R_{\theta\theta}$, one obtains

$$R_{\theta\theta} = 0 \ \Rightarrow \ A - 1 + rA' = 0 \ \Leftrightarrow \ (Ar)' = 1 \ , \tag{11.22}$$

which has the solution $Ar = r + C$ or

$$A(r) = 1 + \frac{C}{r} \ . \tag{11.23}$$

Now also $R_{tt} = R_{rr} = 0$.

To fix C, we compare with the Newtonian limit which tells us that asymptotically $A(r) = -g_{00}$ should approach (temporarily reinserting c) $(1 + 2\Phi/c^2)$, where $\Phi =$

$-GM/r$ is the Newtonian potential for a static spherically symmetric star of mass M. Thus $C = -2MG/c^2$, and the final form of the metric is

$$\boxed{ds^2 = -(1 - \tfrac{2MG}{c^2 r})c^2 dt^2 + (1 - \tfrac{2MG}{c^2 r})^{-1} dr^2 + r^2 d\Omega^2} \tag{11.24}$$

This is the famous *Schwarzschild metric*, obtained by the astronomer Karl Schwarzschild in 1916, the very same year that Einstein published his field equations, while he was serving as a soldier in World War I (and discovered independently a few months later by Johannes Droste, a student of Lorentz).

We will usually not write the constant G explicitly (and set $c = 1$), and thus we introduce the abbreviation

$$m = \frac{GM}{c^2} \quad , \tag{11.25}$$

in terms of which the Schwarzschild metric takes the form

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1} dr^2 + r^2 d\Omega^2 \quad . \tag{11.26}$$

The interpretation of m is that of the *gravitational mass radius* associated to the mass M. To see that all this is dimensionally correct, note that Newton's constant has dimensions (M mass, L length, T time) $[G] = \mathsf{M}^{-1}\mathsf{L}^3\mathsf{T}^{-2}$ so that

$$[G] = \mathsf{M}^{-1}\mathsf{L}^3\mathsf{T}^{-2} \quad \Rightarrow \quad [m] = [GM/c^2] = \mathsf{L} \quad . \tag{11.27}$$

For examples of the value of m for various objects see section 11.5.

We have seen that, by imposing appropriate symmetry conditions on the metric, and making judicious use of them in the course of the calculation, the complicated Einstein equations become rather simple and manageable.

Before discussing some of the remarkable properties of the solution we have just found, I want to mention that the coordinate transformation

$$r = \rho(1 + \frac{MG}{2\rho})^2 \tag{11.28}$$

puts the Schwarzschild metric into isotropic form,

$$ds^2 = -\frac{(1 - \frac{MG}{2\rho})^2}{(1 + \frac{MG}{2\rho})^2} dt^2 + (1 + \frac{MG}{2\rho})^4 (d\rho^2 + \rho^2 d\Omega^2) \quad . \tag{11.29}$$

The advantage of this istropic form of the metric is that one can replace $d\rho^2 + \rho^2 d\Omega^2$ by e.g. the standard metric on $\mathbb{R}^3$ in Cartesian coordinates, or any other metric on $\mathbb{R}^3$. This is useful when one likes to think of the solar system as being essentially described by flat space, with some choice of coordinates.

A slightly more general calculation than we have performed above provides us with (a) somewhat more insight into the interpretation of the parameter M as the mass of the solution and, as an added benefit, (b) a proof of Birkhoff's theorem, mentioned above, that a spherically symmetric vacuum solution of the Einstein equations is necessarily static. Thus, let us start with a general spherically symmetric but not necessarily time-independent metric. Arguments analogous to those leading to (11.5) allow one to conclude that the metric can be chosen to be of the form

$$ds^2 = -A(t,r)dt^2 + B(t,r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{11.30}$$

However, as we have already seen in the derivation of the Schwarzschild metric, this parametrisation of the metric in terms of the two functions $A(r)$ and $B(r)$ is not ideal. To see what might be more convenient, we first reanalyse the Einstein equations in the time-independent case, but this time with an energy-momentum tensor. Thanks to the relation (11.19) we have

$$R^r_{\ r} - R^t_{\ t} = (rB)^{-1}\frac{(AB)'}{AB} \ . \tag{11.31}$$

This suggests that it is useful to introduce a new function $h(r)$ through

$$A(r)B(r) = \mathrm{e}^{2h(r)} \ , \tag{11.32}$$

i.e. through

$$A(r) = \mathrm{e}^{2h(r)}f(r) \quad , \quad B(r) = f(r)^{-1} \tag{11.33}$$

for some arbitrary new function $f(r)$. Using the Einstein equations (9.33) in the form

$$R^\alpha_{\ \beta} = 8\pi G(T^\alpha_{\ \beta} - \tfrac{1}{2}\delta^\alpha_{\ \beta}T^\gamma_{\ \gamma}) \ , \tag{11.34}$$

one sees that one particular linear combination of the Einstein equations now takes the form

$$h'(r) = 4\pi G\, rf(r)^{-1}(T^r_{\ r} - T^t_{\ t}) \ . \tag{11.35}$$

The remaining independent component (in the time-independent case) can be chosen to be

$$R^t_{\ t} - \tfrac{1}{2}R = 8\pi G\, T^t_{\ t} \tag{11.36}$$

which, after a bit of algebra with the formulae (11.18), works out to be

$$[r(f(r)-1)]' = 8\pi G\, r^2 T^t_{\ t} \ . \tag{11.37}$$

This suggests that one trades $f(r)$ for another function

$$r(f(r)-1) = -2m(r) \quad \Leftrightarrow \quad f(r) = 1 - 2m(r)/r \tag{11.38}$$

so that (11.37) becomes

$$m'(r) = 4\pi G\, r^2(-T^t_{\ t}) \ . \tag{11.39}$$

Equations (11.35) and (11.39) make it as manifest as possible that the spherically symmetric vacuum solution is the Schwarzschild metric with $f(r) = 1 - 2m/r$, m constant, and $h(r) = 0$ (an arbitrary constant can be absorbed into the definition of the time-coordinate t).

Let us therefore now, in the time-dependent case, and with the benefit of the above hindsight, parametrise the two arbitrary functions $A(t, r)$ and $B(t, r)$ in (11.30) in terms of two other functions $h(t, r)$ and either $f(t, r)$ or $m(t, r)$ by the substitution

$$\begin{aligned} A(t,r) &= \mathrm{e}^{2h(t,r)} f(t,r) \qquad B(t,r) = f(t,r)^{-1} \\ f(t,r) &= 1 - \frac{2m(t,r)}{r} \quad . \end{aligned} \tag{11.40}$$

In this gauge, the full (non-vacuum) Einstein equations turn out to take a particuarly simple and useful form. The previously obtained equations (11.35) and (11.39) continue to be valid also in the time-dependent case, and there is now one more independent equation, arising from, say, the (rt)-component of the Einstein equation. This set of three equations is (a dot now denotes a t-derivative, a prime as before an r-derivative)

$$\begin{aligned} m'(t,r) &= 4\pi G\, r^2(-T^t_{\ t}) \\ \dot{m}(t,r) &= 4\pi G\, r^2(-T^r_{\ t}) \\ h'(t,r) &= 4\pi G\, r f(t,r)^{-1}(-T^t_{\ t} + T^r_{\ r}) \quad . \end{aligned} \tag{11.41}$$

For vacuum solutions these equations immediately imply that the mass function $m(t, r) = m$ is a constant and that $h'(t, r) = 0$ so that $h = h(t)$ is an arbitrary function of t. Thus h, which only appears in the (tt)-component of the metric, can simply be absorbed into a redefinition of t and we can, without loss of generality, assume that $h = 0$. Thus we uniquely recover the Schwarzschild solution, even without having to assume from the outset that the metric is time-independent. This is Birkhoff's theorem.

Moreover, since $T^t_{\ t}$ is (minus) the energy density and $T^r_{\ t}$ represents the radial energy flux, the above equations show that $m(t, r)$ can inded be interpreted as the mass or energy of the solution.

Finally, note that the characteristic form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \tag{11.42}$$

of the Schwarzschild solution is implied not just by the vacuum Einstein equations but, more generally, by the Einstein equations with $T^t_{\ t} = T^r_{\ r}$. This situation is not as uncommon as one may think. For example, solutions of the Einstein-Maxwell equations for spherically symmetric electrically charged stars, even with the inclusion of a cosmological constant, turn out to also be of this form.[2]

[2] For some further reflections on the ubiquity of such solutions, see T. Jacobson, *When is* $g_{tt}g_{rr} = -1$*?*, `arXiv:0707.3222v3 [gr-qc]`.

11.5 Basic Properties of the Schwarzschild Metric - the Schwarzschild Radius

The metric we have obtained is quite remarkable in several respects. As mentioned before, the vacuum Einstein equations imply that an isotropic metric is static. Furthermore, the metric contains only a single constant of integration, the mass M. This implies that the metric in the exterior of a spherical body is completely independent of the composition of that body. Whatever the energy-momentum tensor for a star may be, the field in the exterior of the star has always got the form (11.24). This considerably simplifies the physical interpretation of General Relativity. In particular, in the subsequent discussion of tests of General Relativity, which only involve the exterior of stars like the sun, we do not have to worry about solutions for the interior of the star and how those could be patched to the exterior solutions.

Let us take a look at the range of coordinates in the Schwarzschild metric. Clearly, t is unrestricted, $-\infty < t < \infty$, and the polar coordinates θ and ϕ have their standard range. However, the issue regarding r is more interesting. First of all, the metric is a vacuum metric. Thus, if the star has radius r_0, then the solution is only valid for $r > r_0$. However, (11.26) also shows that the metric appears to have a singularity at the *Schwarzschild radius* r_S, given by

$$r_S = \frac{2GM}{c^2} = 2m \quad . \tag{11.43}$$

Thus, for the time being we will also require $r > r_S$. Now, in practice the radius of a physical object is almost always much larger than its Schwarzschild radius. For example, for a proton, for the earth and for the sun one has approximately

$$\begin{aligned} M_{\text{proton}} &\sim 10^{-24}\text{ g} \quad \Rightarrow r_S \sim 2,5 \times 10^{-52}\text{ cm} \ll r_0 \sim 10^{-13}\text{cm} \\ M_{\text{earth}} &\sim 6 \times 10^{27}\text{ g} \quad \Rightarrow r_S \sim 1\text{ cm} \ll r_0 \sim 6000\text{km} \\ M_{\text{sun}} &\sim 2 \times 10^{33}\text{ g} \quad \Rightarrow r_S \sim 3\text{ km} \ll r_0 \sim 7 \times 10^5\text{km} \quad . \end{aligned} \tag{11.44}$$

However, for more compact objects, their radius can approach that of their Schwarzschild radius. For example, for neutron stars one can have $r_S \sim 0.1 r_0$, and it is an interesting question (we will take up again later on) what happens to an object whose size is equal to or smaller than its Schwarzschild radius.

One thing that does *not* occur at r_S, however, in spite of what (11.26) may suggest, is a singularity. The singularity in (11.26) is a pure coordinate singularity, an artefact of having chosen a poor coordinate system. One can already see from the metric in isotropic form that in these new coordinates there is no singularity at the Schwarzschild radius, given by $\rho = MG/2$ in the new coordinates. It is true that g_{00} vanishes at that point, but we will later on construct coordinates in which the metric is completely regular at r_S. The only true singularity of the Schwarzschild metric is at $r = 0$, but there the solution was not meant to be valid anyway, so this is not a problem. Nevertheless, as we will see later, something interesting does happen at $r = r_S$, even though there is no singularity and e.g. geodesics are perfectly well behaved there: r_S is an event-horizon, in a sense a point of no return. Once one has passed the Schwarzschild radius of an object with $r_0 < r_S$, there is no turning back, not on geodesics, but also not with any amount of acceleration.

11.6 Measuring Length and Time in the Schwarzschild Metric

In order to learn how to visualise the Schwarzschild metric (for $r > r_0 > r_S$), we will discuss some basic properties of length and time in the Schwarzschild geometry.

Let us first consider proper time for a stationary observer, i.e. an observer at rest at fixed values of (r, θ, ϕ). Proper time is related to coordinate time by

$$d\tau = (1 - 2m/r)^{1/2} dt < dt \quad . \tag{11.45}$$

Thus clocks go slower in a gravitational field - something we already saw in the discussion of the gravitational red-shift, and also in the discussion of the so-called 'twin-paradox': it is this fact that makes the accelerating twin younger than his unaccelerating brother whose proper time would be dt. This formula again suggests that something interesting is happening at the Schwarzschild radius $r = 2m$ - we will come back to this below.

As regards spatial length measurements, thus $dt = 0$, we have already seen above that the slices $r = const.$ have the standard two-sphere geometry. However, as r varies, these two-spheres vary in a way different to the way concentric two-spheres vary in $\mathbb{R}^3$. To see this, note that the proper radius R, obtained from the spatial line element by setting $\theta = const., \phi = const$, is

$$dR = (1 - 2m/r)^{-1/2} dr > dr \quad . \tag{11.46}$$

In other words, the proper radial distance between concentric spheres of area $4\pi r^2$ and area $4\pi(r + dr)^2$ is $dR > dr$ and hence larger than in flat space. Note that $dR \to dr$ for $r \to \infty$ so that, as expected, far away from the origin the space approximately looks like $\mathbb{R}^3$. One way to visualise this geometry is as a sort of throat or sink, as in Figure 9.

To get some more quantitative feeling for the distortion of the geometry produced by the gravitational field of a star, consider a long stick lying radially in this gravitational field, with its endpoints at the coordinate values $r_1 > r_2$. To compute its length L, we have to evaluate

$$L = \int_{r_2}^{r_1} dr (1 - 2m/r)^{-1/2} \quad . \tag{11.47}$$

It is possible to evaluate this integral in closed form (by changing variables from r to $u = 1/r$), but for the present purposes it will be enough to treat $2m/r$ as a small

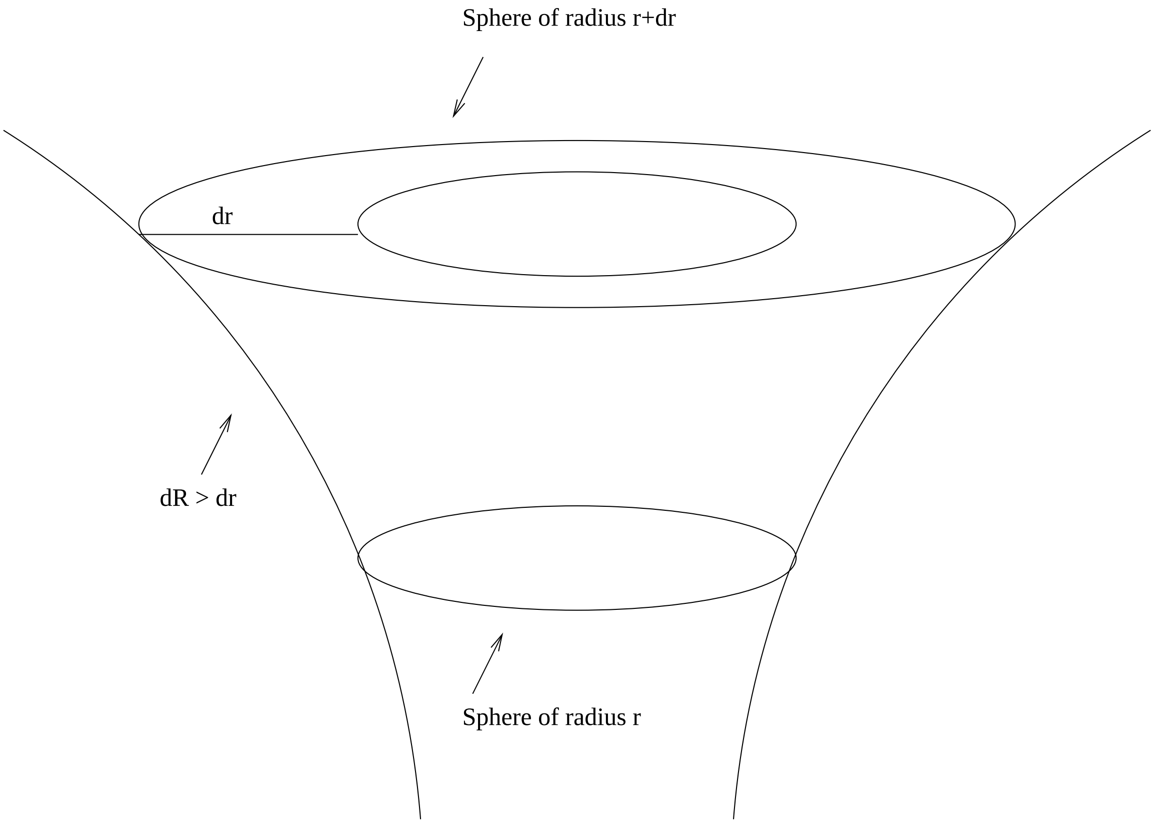

Figure 9: Figure illustrating the geometry of the Schwarzschild metric. In $\mathbb{R}^3$, concentric spheres of radii r and $r + dr$ are a distance dr apart. In the Schwarzschild geometry, such spheres are a distance $dR > dr$ apart. This departure from Euclidean geometry becomes more and more pronounced for smaller values of r, i.e. as one travels down the throat towards the Schwarzschild radius $r = 2m$.

perturbation and to only retain the term linear in m in the Taylor expansion. Then we find

$$L \approx \int_{r_2}^{r_1} dr(1 + m/r) = (r_1 - r_2) + m \log \frac{r_1}{r_2} > (r_1 - r_2) \ . \tag{11.48}$$

We see that the corrections to the Euclidean result are suppressed by powers of the Schwarzschild radius $r_S = 2m$ so that for most astronomical purposes one can simply work with coordinate distances!

12 Particle and Photon Orbits in the Schwarzschild Geometry

We now come to the heart of the matter, the study of planetary orbits and light rays in the gravitational field of the sun, i.e. the properties of timelike and null geodesics of the Schwarzschild geometry. We shall see that, by once again making good use of the symmetries of the problem, we can reduce the geodesic equations to a single first order differential equation in one variable, analogous to that for a one-dimensional particle moving in a particular potential. Solutions to this equation can then readily be discussed qualitatively and also quantitatively (analytically).

12.1 From Conserved Quantities to the Effective Potential

A convenient starting point in general for discussing geodesics is, as I stressed before, the Lagrangian $\mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$. For the Schwarzschild metric this is

$$\mathcal{L} = -(1-2m/r)\dot{t}^2 + (1-2m/r)^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \ , \tag{12.1}$$

where $2m = 2MG/c^2$. Rather than writing down and solving the (second order) geodesic equations, we will make use of the conserved quantities $K_\mu \dot{x}^\mu$ associated with Killing vectors. After all, conserved quantities correspond to first integrals of the equations of motion and if there are a sufficient number of them (there are) we can directly reduce the second order differential equations to first order equations.

So, how many Killing vectors does the Schwarzschild metric have? Well, since the metric is static, there is one timelike Killing vector, namely $\partial/\partial t$, and since the metric is spherically symmetric, there are spatial Killing vectors generating the Lie algebra of $SO(3)$, hence there are three of those, and therefore all in all four Killing vectors.

Now, since the gravitational field is isotropic (and hence there is conservation of angular momentum), the orbits of the particles or planets are planar. Without loss of generality, we can choose our coordinates in such a way that this plane is the equatorial plane $\theta = \pi/2$, so in particular $\dot{\theta} = 0$, and the residual Lagrangian to deal with is

$$\mathcal{L}' = -(1-2m/r)\dot{t}^2 + (1-2m/r)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 \ . \tag{12.2}$$

This choice fixes the direction of the angular momentum (to be orthogonal to the plane) and leaves two conserved quantities, the energy (per unit rest mass) E and the magnitude L (per unit rest mass) of the angular momentum, corresponding to the cyclic variables t and ϕ, (or: corresponding to the Killing vectors $\partial/\partial t$ and $\partial/\partial\phi$),

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial t} = 0 \quad &\Rightarrow \quad \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{t}} = 0 \\ \frac{\partial\mathcal{L}}{\partial\phi} = 0 \quad &\Rightarrow \quad \frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = 0 \ , \end{aligned} \tag{12.3}$$

namely

$$E = (1-2m/r)\dot{t} \tag{12.4}$$
$$L = r^2\sin^2\theta\dot{\phi} = r^2\dot{\phi} \ . \tag{12.5}$$

Calling L the angular momentum (per unit rest mass) requires no further justification, but let me pause to explain in what sense E is an energy (per unit rest mass). On the one hand, it is the conserved quantity (2.45) associated to time-translation invariance. As such, it certainly deserves to be called the energy.

But it is moreoever true that for a particle at infinity ($r \to \infty$) E is just the special relativistic energy $E = \gamma(v_\infty)c^2$, with $\gamma(v) = (1-v^2/c^2)^{-1/2}$ the usual relativistic γ-factor, and v_∞ the coordinate velocity dr/dt at infinity. This can be seen in two ways. First of all, for a particle that reaches $r = \infty$, the constant E can be determined by evaluating it at $r = \infty$. It thus follows from the definition of E that

$$E = \dot{t}_\infty \ . \tag{12.6}$$

In Special Relativity, the relation between proper and coordinate time is given by (setting $c = 1$ again)

$$d\tau = \sqrt{1-v^2}dt \quad \Rightarrow \quad \dot{t} = \gamma(v) \ , \tag{12.7}$$

suggesting the identification

$$E = \gamma(v_\infty) \qquad (E = \gamma(v_\infty)c^2 \quad \text{if} \quad c \neq 1) \tag{12.8}$$

Another argument for this identification will be given below, once we have introduced the effective potential.

As we have seen in section 2.1 (and again in section 4.8), there is also always one more integral of the geodesic equation (corresponding roughly speaking to parametrisation invariance of the Lagrangian), namely $\mathcal{L}$ itself,

$$\frac{d}{d\tau}\mathcal{L} = 2g_{\mu\nu}\dot{x}^\mu\frac{D}{D\tau}\dot{x}^\nu = 0 \ . \tag{12.9}$$

Thus we set

$$\mathcal{L} = \epsilon \ , \tag{12.10}$$

where $\epsilon = -1$ for timelike geodesics and $\epsilon = 0$ for null geodesics. Thus we have

$$-(1-2m/r)\dot{t}^2 + (1-2m/r)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = \epsilon \ , \tag{12.11}$$

and we can now express $\dot{t}$ and $\dot{\phi}$ in terms of the conserved quantities E and L to obtain a first order differential equation for r alone, namely

$$-(1-2m/r)^{-1}E^2 + (1-2m/r)^{-1}\dot{r}^2 + \frac{L^2}{r^2} = \epsilon \ . \tag{12.12}$$

Multiplying by $(1-2m/r)/2$ and rearranging the terms, one obtains

$$\frac{E^2+\epsilon}{2} = \frac{\dot{r}^2}{2} + \epsilon\frac{m}{r} + \frac{L^2}{2r^2} - \frac{mL^2}{r^3} \quad . \tag{12.13}$$

Now this equation is of the familiar Newtonian form

$$E_{eff} = \frac{\dot{r}^2}{2} + V_{eff}(r) \quad , \tag{12.14}$$

with

$$\begin{aligned} E_{eff} &= \frac{E^2+\epsilon}{2} \\ V_{eff}(r) &= \epsilon\frac{m}{r} + \frac{L^2}{2r^2} - \frac{mL^2}{r^3} \quad , \end{aligned} \tag{12.15}$$

describing the energy conservation in an effective potential. Except for $t \to \tau$, this is exactly the same as the Newtonian equation of motion in a potential

$$V(r) = \epsilon\frac{m}{r} - \frac{mL^2}{r^3} \quad , \tag{12.16}$$

the effective angular momentum term $L^2/r^2 = r^2\dot{\phi}^2$ arising, as usual, from the change to polar coordinates.

In particular, for $\epsilon = -1$, the general relativistic motion (as a function of τ) is exactly the same as the Newtonian motion (as a function of t) in the potential

$$\epsilon = -1 \quad \Rightarrow \quad V(r) = -\frac{m}{r} - \frac{mL^2}{r^3} \quad . \tag{12.17}$$

The first term is just the ordinary Newtonian potential, so the second term is apparently a general relativistic correction. We will later on treat this as a perturbation but note that the above is an exact result, not an approximation (so, for example, there are no higher order corrections proportional to higher powers of m/r). We expect observable consequences of this general relativistic correction because many properties of the Newtonian orbits (Kepler's laws) depend sensitively on the fact that the Newtonian potential is precisely $\sim 1/r$.

Looking at the equation for $\epsilon = -1$,

$$\tfrac{1}{2}\dot{r}^2 + V_{eff}(r) = \tfrac{1}{2}(E^2-1) \tag{12.18}$$

and noting that $V_{eff}(r) \to 0$ for $r \to \infty$, we can read off that for a particle that reaches $r = \infty$ we have the relation

$$\dot{r}_\infty^2 = E^2 - 1 \quad . \tag{12.19}$$

This implies, in particular, that for such (scattering) trajectories one necessarily has $E \geq 1$, with $E = 1$ corresponding to a particle initially or finally at rest at infinity. For $E > 1$ the coordinate velocity at infinity can be computed from

$$v_\infty^2 = \frac{\dot{r}_\infty^2}{\dot{t}_\infty^2} \quad . \tag{12.20}$$

Using (12.6) and (12.19), one finds

$$v_\infty^2 = \frac{E^2 - 1}{E^2} \quad \Leftrightarrow \quad E = (1 - v_\infty^2)^{-1/2} \ , \tag{12.21}$$

thus confirming the result claimed in (12.8).

For null geodesics, on the other hand, the Newtonian part of the potential is zero, as one might expect for massless particles, but in General Relativity a photon feels a non-trivial potential

$$\epsilon = 0 \ \Rightarrow \ V(r) = -\frac{mL^2}{r^3} \ . \tag{12.22}$$

12.2 The Equation for the Shape of the Orbit

Typically, one is primarily interested in the shape of an orbit, that is in the radius r as a function of ϕ, $r = r(\phi)$, rather than in the dependence of, say, r on some extraterrestrial's proper time τ. In this case, the above mentioned difference between t (in the Newtonian theory) and τ (here) is irrelevant: In the Newtonian theory one uses $L = r^2 d\phi/dt$ to express t as a function of ϕ, $t = t(\phi)$ to obtain $r(\phi)$ from $r(t)$. In General Relativity, one uses the analogous equation $L = r^2 d\phi/d\tau$ to express τ as a function of ϕ, $\tau = \tau(\phi)$. Hence the shapes of the General Relativity orbits are precisely the shapes of the Newtonian orbits in the potential (12.16). Thus we can use the standard methods of Classical Mechanics to discuss these general relativistic orbits and of course this simplifies matters considerably.

To obtain r as a function of ϕ we proceed as indicated above. Thus we use

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{\dot{r}^2}{\dot{\phi}^2} \tag{12.23}$$

to combine (12.14),

$$\dot{r}^2 = 2E_{eff} - 2V_{eff}(r) \ , \tag{12.24}$$

and (12.5),

$$\dot{\phi}^2 = \frac{L^2}{r^4} \tag{12.25}$$

into

$$\frac{r'^2}{r^4} L^2 = 2E_{eff} - 2V_{eff}(r) \tag{12.26}$$

where a prime denotes a ϕ-derivative.

In the examples to be discussed below, we will be interested in the angle $\Delta\phi$ swept out by the object in question (a planet or a photon) as it travels along its trajectory between the farthest distance r_2 from the star (sun) ($r_2 = \infty$ for scattering trajectories) and the position of closest approach to the star r_1 (the perihelion or, more generally, if we are not talking about our own solar system, periastron), and back again,

$$\Delta\phi = 2 \int_{r_1}^{r_2} \frac{d\phi}{dr} dr \ . \tag{12.27}$$

In the Newtonian case, these integrals can be evaluated in closed form. With the general relativistic correction term, however, these are elliptic integrals which can not be expressed in closed form. A perturbative evaluation of these integrals (treating the exact general relativistic correction as a small perturbation) also turns out to be somewhat delicate since e.g. the limits of integration depend on the perturbation.

It is somewhat simpler to deal with this correction term not at the level of the solution (integral) but at the level of the corresponding differential equation. As in the Kepler problem, it is convenient to make the change of variables

$$u = \frac{1}{r} \qquad u' = -\frac{r'}{r^2} \quad . \tag{12.28}$$

Then (12.26) becomes

$$u'^2 = L^{-2}(2E_{eff} - 2V_{eff}(r)) \quad . \tag{12.29}$$

Upon inserting the explicit expression for the effective potential, this becomes

$$u'^2 + u^2 = \frac{E^2 + \epsilon}{L^2} - \frac{2\epsilon m}{L^2}u + 2mu^3 \quad . \tag{12.30}$$

This can be used to obtain an equation for $d\phi(u)/du = u'^{-1}$, leading to

$$\Delta\phi = 2\int_{u_2}^{u_1} \frac{d\phi}{du} du \quad . \tag{12.31}$$

Differentiating (12.30) once more, one finds

$$u'(u'' + u) = u'(-\frac{\epsilon m}{L^2} + 3mu^2) \quad . \tag{12.32}$$

Thus either $u' = 0$, which corresponds to a circular, constant radius, orbit and is irrelevant since neither the planets nor the photons of interest to us travel on circular orbits, or

$$u'' + u = -\frac{\epsilon m}{L^2} + 3mu^2 \quad . \tag{12.33}$$

This is the equation that we will study below to determine the perihelion shift and the bending of light by a star. In the latter case, which is a bit simpler, I will also sketch two other derivations of the result, based on different perturbative evaluations of the elliptic integral.

12.3 Timelike Geodesics

We will first try to gain a qualititative understanding of the behaviour of geodesics in the effective potential

$$V_{eff}(r) = -\frac{m}{r} + \frac{L^2}{2r^2} - \frac{mL^2}{r^3} \quad . \tag{12.34}$$

The standard way to do this is to plot this potential as a function of r for various values of the parameters L and m. The basic properties of $V_{eff}(r)$ are the following:

1. Asymptotically, i.e. for $r \to \infty$, the potential tends to the Newtonian potential,

$$V_{eff}(r) \stackrel{r\to\infty}{\longrightarrow} -\frac{m}{r} \quad . \tag{12.35}$$

2. At the Schwarzschild radius $r_S = 2m$, nothing special happens and the potential is completely regular there,

$$V_{eff}(r = 2m) = -\frac{1}{2} \quad . \tag{12.36}$$

   For the discussion of planetary orbits in the solar system we can safely assume that the radius of the sun is much larger than its Schwarzschild radius, $r_0 \gg r_S$, but the above shows that even for these highly compact objects with $r_0 < r_S$ geodesics are perfectly regular as one approaches r_S. Of course the particular numerical value of $V_{eff}(r = 2m)$ has no special significance because $V(r)$ can always be shifted by a constant.

3. The extrema of the potential, i.e. the points at which $dV_{eff}/dr = 0$, are at

$$r_\pm = (L^2/2m)[1 \pm \sqrt{1 - 12(m/L)^2}] \quad , \tag{12.37}$$

   and the potential has a maximum at r_- and a local minimum at r_+. Thus there are qualitative differences in the shapes of the orbits between $L/m < \sqrt{12}$ and $L/m > \sqrt{12}$.

Let us discuss these two cases in turn. When $L/m < \sqrt{12}$, then there are no critical points and the potential looks approximately like that in Figure 10. Note that we should be careful with extrapolating to values of r with $r < 2m$ because we know that the Schwarzschild metric has a coordinate singularity there. However, qualitatively the picture is also correct for $r < 2m$.

From this picture we can read off that there are no bounded orbits for these values of the parameters. Any inward bound particle with $L < \sqrt{12}m$ will continue to fall inwards (provided that it moves on a geodesic). This should be contrasted with the Newtonian situation in which for any $L \neq 0$ there is always the centrifugal barrier reflecting incoming particles since the repulsive term $L^2/2r^2$ will dominate over the attractive $-m/r$ for small values of r. In General Relativity, on the other hand, it is the attractive term $-mL^2/r^3$ that dominates for small r.

Fortunately for the stability of the solar system, the situation is qualitatively quite different for sufficiently large values of the angular momentum, namely $L > \sqrt{12}m$ (see Figure 11).

In that case, there is a minimum and a maximum of the potential. The critical radii correspond to exactly circular orbits, unstable at r_- (on top of the potential) and stable at r_+ (the minimum of the potential). For $L \to \sqrt{12}m$ these two orbits approach each

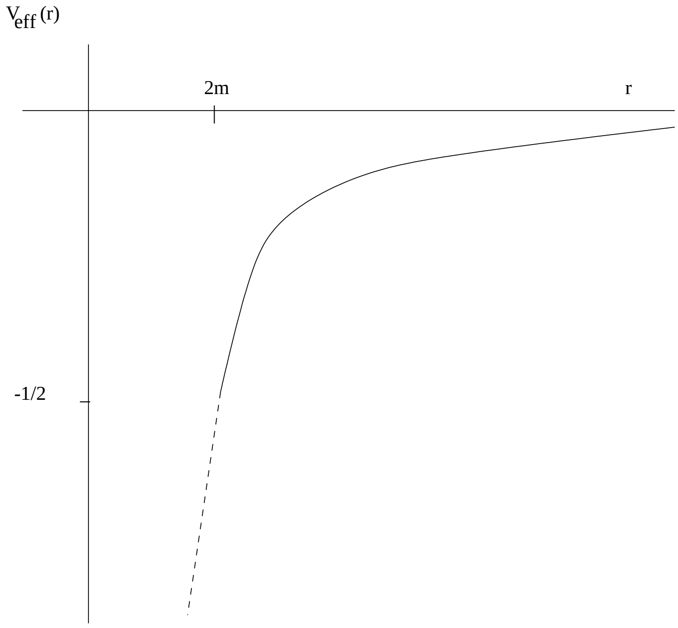

Figure 10: Effective potential for a massive particle with $L/m < \sqrt{12}$. The extrapolation to values of $r < 2m$ has been indicated by a dashed line.

other, the critical radius tending to $r_\pm \to 6m$. Thus the innermost stable circular orbit (known affectionately as the ISCO in astrophysics) is located at

$$r_{ISCO} = 6m \ . \tag{12.38}$$

On the other hand, for very large values of L the critical radii are (expand the square root to first order) to be found at

$$(r_+, r_-) \stackrel{L\to\infty}{\longrightarrow} (L^2/m, 3m) \ . \tag{12.39}$$

For given L, for sufficiently large values of E_{eff} a particle will fall all the way down the potential. For $E_{eff} < 0$, there are bound orbits which are not circular and which range between the radii r_1 and r_2, the turning points at which $\dot{r} = 0$ and therefore $E_{eff} = V_{eff}(r)$.

12.4 The Anomalous Precession of the Perihelia of the Planetary Orbits

Because of the general relativistic correction $\sim 1/r^3$, the bound orbits will not be closed (elliptical). In particular, the position of the perihelion, the point of closest approach of the planet to the sun where the planet has distance r_1, will not remain constant. However, because r_1 is constant, and the planetary orbit is planar, this point will move on a circle of radius r_1 around the sun.

As described in section 12.2, in order to calculate this perihelion shift one needs to calculate the total angle $\Delta\phi$ swept out by the planet during one revolution by integrating

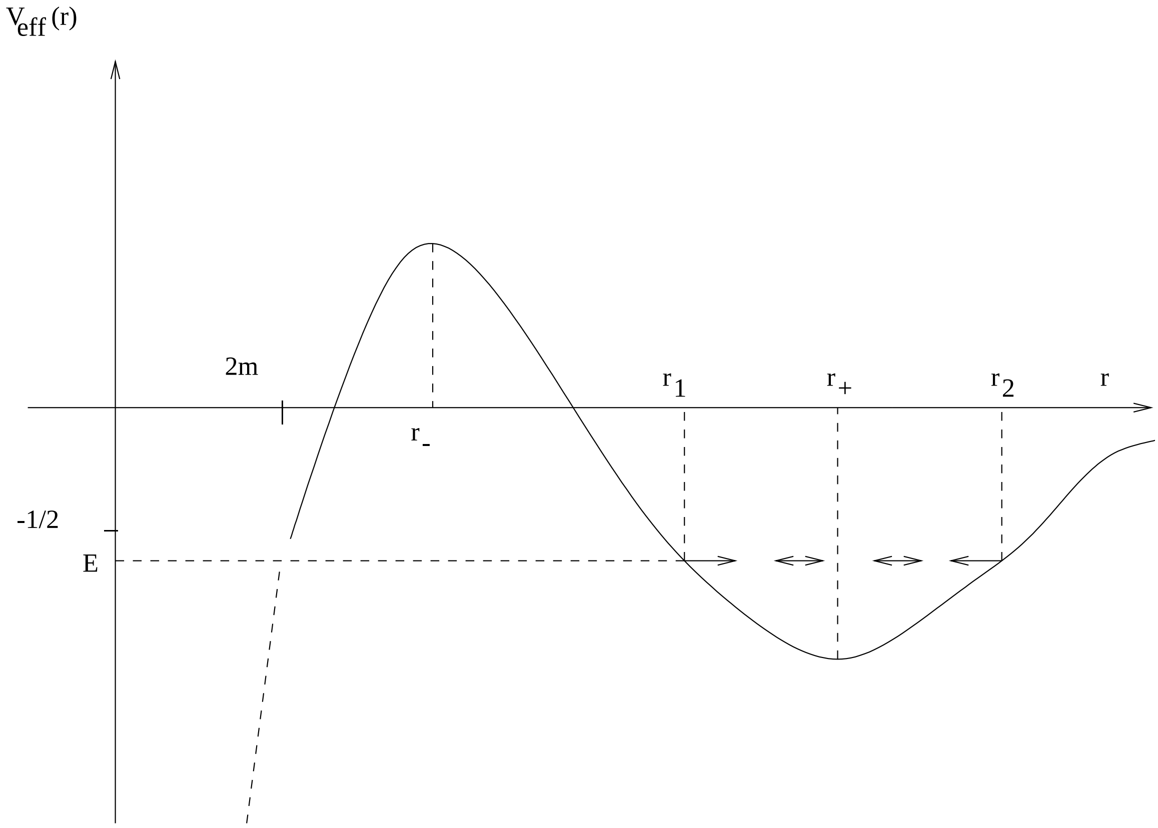

Figure 11: Effective potential for a massive particle with $L/m > \sqrt{12}$. Shown are the maximum of the potential at r_- (an unstable circular orbit), the minimum at r_+ (a stable circular orbit), and the orbit of a particle with $E_{eff} < 0$ with turning points r_1 and r_2.

this from r_1 to r_2 and back again to r_1, or

$$\Delta\phi = 2\int_{r_1}^{r_2} \frac{d\phi}{dr} dr \quad . \tag{12.40}$$

Rather than trying to evaluate the above integral via some sorcery, we will determine $\Delta\phi$ by analysing the orbit equation (12.33) for $\epsilon = -1$,

$$u'' + u = \frac{m}{L^2} + 3mu^2 \quad . \tag{12.41}$$

In the Newtonian approximation, this equation reduces to that of a displaced harmonic oscillator,

$$u_0'' + u_0 = \frac{m}{L^2} \quad \Leftrightarrow \quad (u_0 - m/L^2)'' + (u_0 - m/L^2) = 0 \quad , \tag{12.42}$$

and the solution is a Kepler ellipse described parametrically by

$$u_0(\phi) = \frac{m}{L^2}(1 + e\cos\phi) \tag{12.43}$$

where e is the eccentricity ($e = 0$ means constant radius and hence a circular orbit). Plugging this back into the Newtonian non-linear 1st-order equation (cf. (12.30))

$$u_0'^2 + u_0^2 = \frac{E^2 - 1}{L^2} + \frac{2m}{L^2}u_0 \quad , \tag{12.44}$$

one finds that the integration constant e is related to the energy by

$$e^2 = 1 + \frac{L^2}{m^2}(E^2 - 1) = 1 + \frac{2L^2}{m^2}E_{eff} \quad . \tag{12.45}$$

In particular, $e^2 < 1$ for bound states (bounded orbits) with $E_{eff} < 0$, and we will concentrate on these orbits. The perihelion (aphelion) is then at $\phi = 0$ ($\phi = \pi$), with

$$r_{1,2} = \frac{L^2}{m}\frac{1}{1 \pm e} \quad . \tag{12.46}$$

Thus the semi-major axis a of the ellipse,

$$2a = r_1 + r_2 \quad , \tag{12.47}$$

is

$$a = \frac{L^2}{m}\frac{1}{1 - e^2} \quad . \tag{12.48}$$

In particular, in the Newtonian theory, one has

$$(\Delta\phi)_0 = 2\pi \quad . \tag{12.49}$$

The anomalous perihelion shift due to the effects of General Relativity is thus

$$\delta\phi = \Delta\phi - 2\pi \quad . \tag{12.50}$$

In order to determine $\delta\phi$, we now seek a solution to (12.41) of the form

$$u = u_0 + u_1 \tag{12.51}$$

where u_1 is a small deviation. This leads to the equation

$$u_1'' + u_1 = 3mu_0^2 \ . \tag{12.52}$$

The general solution of this inhomogenous differential equation is the general solution of the homogeneous equation (we are not interested in) plus a special solution of the inhomogeneous equation. Writing

$$(1 + e\cos\phi)^2 = (1 + \tfrac{1}{2}e^2) + 2e\cos\phi + \tfrac{1}{2}e^2\cos 2\phi \tag{12.53}$$

and noting that

$$\begin{aligned} (\phi\sin\phi)'' + \phi\sin\phi &= 2\cos\phi \\ (\cos 2\phi)'' + \cos 2\phi &= -3\cos 2\phi \end{aligned} \tag{12.54}$$

one sees that a special solution is

$$u_1(\phi) = \frac{3m^3}{L^4}((1 + \tfrac{1}{2}e^2) - \tfrac{1}{6}e^2\cos 2\phi + e\phi\sin\phi) \ . \tag{12.55}$$

The term of interest to us is the third term which provides a cumulative non-periodic effect over successive orbits. Focussing on this term, we can write the approximate solution to the orbit equation as

$$u(\phi) \approx \frac{m}{L^2}(1 + e\cos\phi + \frac{3m^2e}{L^2}\phi\sin\phi) \ . \tag{12.56}$$

If the first perihelion is at $\phi = 0$, the next one will be at a point $\Delta\phi = 2\pi + \delta\phi$ close to 2π which is such that $u'(\Delta\phi) = 0$ or

$$\sin\delta\phi = \frac{3m^2}{L^2}(\sin\delta\phi + (2\pi + \delta\phi)\cos\delta\phi) \ . \tag{12.57}$$

Using that $\delta\phi$ is small, and keeping only the lowest order terms in this equation, one finds the result

$$\delta\phi = \frac{6\pi m^2}{L^2} = 6\pi(\frac{GM}{cL})^2 \ . \tag{12.58}$$

An alternative way to obtain this result is to observe that (12.56) can be approximately written as

$$u(\phi) \approx \frac{m}{L^2}\left(1 + e\cos[(1 - \frac{3m^2}{L^2})\phi]\right) \tag{12.59}$$

From this equation it is manifest that during each orbit the perihelion advances by

$$\delta\phi = 2\pi\frac{3m^2}{L^2} \tag{12.60}$$

$(2\pi(1-3m^2/L^2)(1+3m^2/L^2) \approx 2\pi)$ in agreement with the above result.

In terms of the eccentricity e and the semi-major axis a (12.48) of the elliptical orbit, this can be written as

$$\delta\phi = \frac{6\pi G}{c^2}\frac{M}{a(1-e^2)} \quad . \tag{12.61}$$

As these parameters are known for the planetary orbits, $\delta\phi$ can be evaluated. For example, for Mercury, where this effect is largest (because it has the largest eccentricity) one finds $\delta\phi = 0,1''$ per revolution. This is of course a tiny effect (1 second, $1''$, is one degree divided by 3600) and not *per se* detectable. However,

1. this effect is cumulative, i.e. after N revolutions one has an anomalous perihelion shift $N\delta\phi$;
2. Mercury has a very short solar year, with about 415 revolutions per century;
3. and accurate observations of the orbit of Mercury go back over 200 years.

Thus the cumulative effect is approximately $850\delta\phi$ and this *is* sufficiently large to be observable. The prediction of General Relativity for this effect is

$$\delta\phi_{GR} = 43,03''/\text{century} \quad . \tag{12.62}$$

And indeed such an effect is observed (and had for a long time presented a puzzle, an anomaly, for astronomers). In actual fact, the perihelion of Mercury's orbit shows a precession rate of $5601''$ per century, so this does not yet look like a brilliant confirmation of General Relativity. However, of this effect about $5025''$ are due to fact that one is using a non-inertial geocentric coordinate system (precession of the equinoxes). $532''$ are due to perturbations of Mercury's orbit caused by the (Newtonian) gravitational attraction of the other planets of the solar system (chiefly Venus, earth and Jupiter). This much was known prior to General Relativity and left an unexplained anomalous perihelion shift of

$$\delta\phi_{anomalous} = 43,11'' \pm 0,45''/\text{century} \quad . \tag{12.63}$$

Now the agreement with the result of General Relativity is truly impressive and this is one of the most important experimental verifications of General Relativity. Other observations, involving e.g. the mini-planet Icarus, discovered in 1949, with a huge eccentricity $e \sim 0,827$, or binary pulsar systems, have provided further confirmation of the agreement between General Relativity and experiment.

12.5 Null Geodesics

To study the behaviour of massless particles (photons) in the Schwarzschild geometry, we need to study the effective potential

$$V_{eff}(r) = \frac{L^2}{2r^2} - \frac{mL^2}{r^3} = \frac{L^2}{2r^2}(1-\frac{2m}{r}) \quad . \tag{12.64}$$

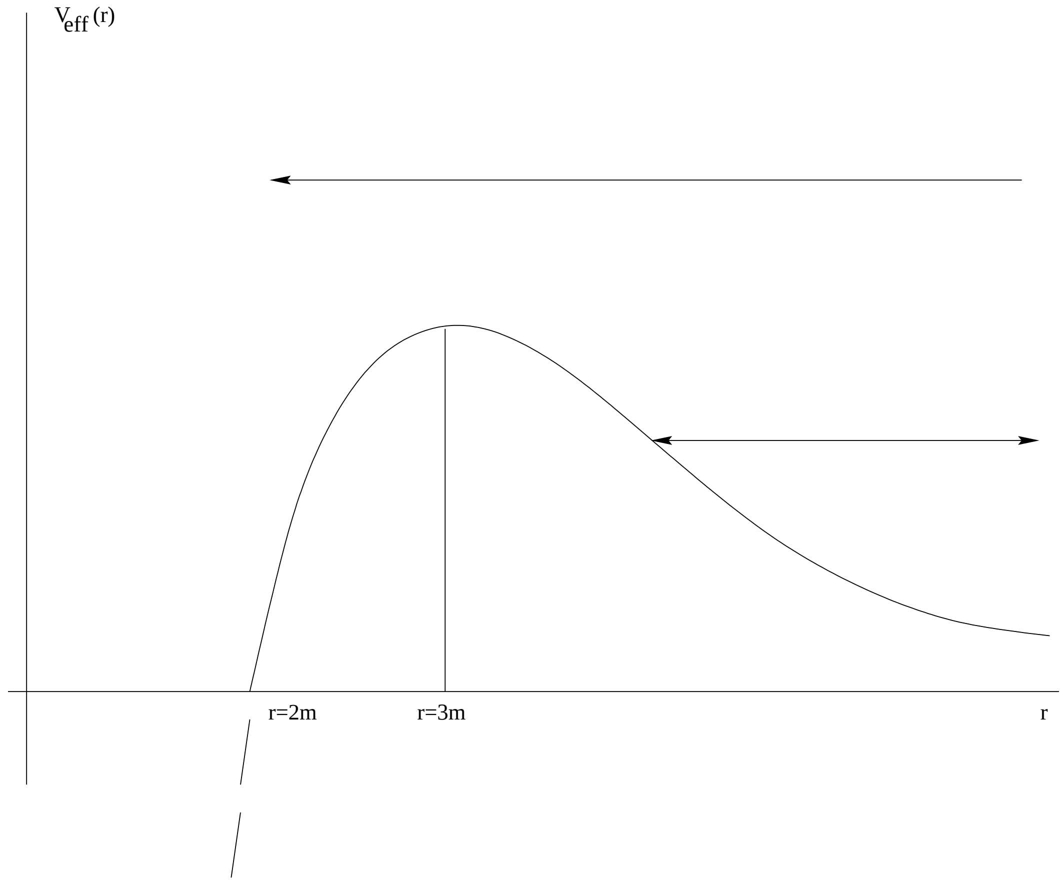

Figure 12: Effective potential for a massless particle. Displayed is the location of the unstable circular orbit at $r = 3m$. A photon with an energy $E^2 < L^2/27m^2$ will be deflected (lower arrow), photons with $E^2 > L^2/27m^2$ will be captured by the star.

The following properties are immediate:

1. For $r > 2m$, the potential is positive, $V(r) > 0$.
2. For $r \to \infty$, one has $V_{eff}(r) \to 0$.
3. $V_{eff}(r = 2m) = 0$.
4. When $L = 0$, the photons feel no potential at all.
5. There is one critical point of the potential, at $r = 3m$, with $V_{eff}(r = 3m) = L^2/54m^2$.

Thus the potential has the form sketched in Figure 12.

For energies $E^2 > L^2/27m^2$, photons are captured by the star and will spiral into it. For energies $E^2 < L^2/27m^2$, on the other hand, there will be a turning point, and light rays will be deflected by the star. As this may sound a bit counterintuitive (shouldn't a photon with higher energy be more likely to zoom by the star without being forced to spiral into it?), think about this in the following way. $L = 0$ corresponds to a photon falling radially towards the star, L small corresponds to a slight deviation from radial motion, while L large (thus $\dot{\phi}$ large) means that the photon is travelling along a trajectory that will not bring it very close to the star at all (see the next subsection for the precise relation between the angular momentum L and the impact parameter b of the photon). It is then not surprising that photons with small L are more likely to be captured by the star (this happens for $L^2 < 27m^2E^2$) than photons with large L which will only be deflected in their path.

We will study this in more detail below. But let us first also consider the opposite situation, that of light from or near the star (and we are of course assuming that $r_0 > r_S$). Then for $r_0 < 3m$ and $E^2 < L^2/27m^2$, the light cannot escape to infinity but falls back to the star, whereas for $E^2 > L^2/27m^2$ light will escape. Thus for a path sufficiently close to radial (L small, because $\dot{\phi}$ is then small) light can always escape as long as $r > 2m$.

The existence of one unstable circular orbit for photons at $r = 3m$ (the *photon sphere*), while not relevant for the applications to the solar system in this section, turns out to be of some interest in black hole astrophysics (as a possibly observable signature of black holes).

12.6 The Bending of Light by a Star: 3 Derivations

To study the bending of light by a star, we consider an incoming photon (or light ray) with impact parameter b (see Figure 13) and we need to calculate $\phi(r)$ for a trajectory with turning point at $r = r_1$. At that point we have $\dot{r} = 0$ (the dot now indicates differentiation with respect to the affine parameter σ of the null geodesic, we can but need not choose this to be the coordinate time t) and therefore r_1 is determined by

$$E_{eff} = V_{eff}(r_1) \quad \Leftrightarrow \quad r_1^2 = \frac{L^2}{E^2}(1 - \frac{2m}{r_1}) \quad . \tag{12.65}$$

The first thing we need to establish is the relation between b and the other parameters E and L. Consider the ratio

$$\frac{L}{E} = \frac{r^2\dot{\phi}}{(1-2m/r)\dot{t}} \quad . \tag{12.66}$$

For large values of r, $r \gg 2m$, this reduces to

$$\frac{L}{E} = r^2\frac{d\phi}{dt} \quad . \tag{12.67}$$

On the other hand, for large r we can approximate $b/r = \sin\phi$ by ϕ. Since we also have $dr/dt = -1$ (for an incoming light ray), we deduce

$$\frac{L}{E} = r^2\frac{d}{dt}\frac{b}{r} = b \quad . \tag{12.68}$$

In terms of the variable $u = 1/r$ the equation for the shape of the orbit (12.33) is

$$u'' + u = 3mu^2 \tag{12.69}$$

and the elliptic integral (12.31) for $\Delta\phi$ is

$$\Delta\phi = 2\int_{r_1}^{\infty} \frac{d\phi}{dr} dr = 2\int_0^{u_1} du\, [b^{-2} - u^2 + 2mu^3]^{-1/2} \ . \tag{12.70}$$

Moreover, in terms of u we can write the equation (12.65) for $u_1 = 1/r_1$ as

$$b^{-2} = u_1^2 - 2mu_1^3 \ . \tag{12.71}$$

In the absence of the general relativistic correction (calling this 'Newtonian' is perhaps not really appropriate since we are dealing with photons/light rays) one has $b^{-1} = u_1$ or $b = r_1$ (no deflection). The orbit equation

$$u_0'' + u_0 = 0 \tag{12.72}$$

has the solution

$$u_0(\phi) = \frac{1}{b}\sin\phi \ , \tag{12.73}$$

describing the straight line

$$r_0(\phi)\sin\phi = b \ . \tag{12.74}$$

Obligingly the integral gives

$$(\Delta\phi)_0 = 2\int_0^1 dx(1-x^2)^{-1/2} = 2\arcsin 1 = \pi \ . \tag{12.75}$$

Thus the deflection angle is related to $\Delta\phi$ by

$$\delta\phi = \Delta\phi - \pi \ . \tag{12.76}$$

We will now determine $\delta\phi$ in three different ways,

- by perturbatively solving the orbit equation (12.69);
- by perturbatively evaluating the elliptic integral (12.70);
- by performing a perturbative expansion (linearisation) of the Schwarzschild metric.

Derivation I: Perturbative Solution of the Orbit Equation

In order to solve the orbit equation (12.69), we proceed as in section 12.4. Thus the equation for the (small) deviation $u_1(\phi)$ is

$$u_1'' + u_1 = 3mu_0^2 = \frac{3m}{b^2}(1-\cos^2\phi) = \frac{3m}{2b^2}(1-\cos 2\phi) \tag{12.77}$$

which has the particular solution (cf. (12.54))

$$u_1(\phi) = \frac{3m}{2b^2}(1 + \tfrac{1}{3}\cos 2\phi) \quad . \tag{12.78}$$

Therefore

$$u(\phi) = \frac{1}{b}\sin\phi + \frac{3m}{2b^2}(1 + \tfrac{1}{3}\cos 2\phi) \quad . \tag{12.79}$$

By considering the behaviour of this equation as $r \to \infty$ or $u \to 0$, one finds an equation for (minus) half the deflection angle, namely

$$\frac{1}{b}(-\delta\phi/2) + \frac{3m}{2b^2}\frac{4}{3} = 0 \quad , \tag{12.80}$$

leading to the result

$$\delta\phi = \frac{4m}{b} = \frac{4MG}{bc^2} \quad . \tag{12.81}$$

Derivation II: Perturbative Evaluation of the Elliptic Integral

The perturbative evaluation of (12.70) is rather tricky when it is regarded as a function of the independent variables m and b, with r_1 determined by (12.65) (try this!). The trick to evaluate (12.70) is (see R. Wald, *General Relativity*) to regard the integral as a function of the independent variables r_1 and m, with b eliminated via (12.71). Thus (12.70) becomes

$$\Delta\phi = 2\int_0^{u_1} du\; [u_1^2 - u^2 - 2m(u_1^3 - u^3)]^{-1/2} \quad . \tag{12.82}$$

The first order correction

$$\Delta\phi = (\Delta\phi)_0 + m(\Delta\phi)_1 + \mathcal{O}(m^2) \tag{12.83}$$

is therefore

$$(\Delta\phi)_1 = \left(\frac{\partial}{\partial m}\Delta\phi\right)_{m=0} = 2\int_0^{b^{-1}} du\; \frac{b^{-3} - u^3}{(b^{-2} - u^2)^{3/2}} \quad . \tag{12.84}$$

This integral is elementary,

$$\int dx\; \frac{1 - x^3}{(1 - x^2)^{3/2}} = -(x + 2)\left(\frac{1 - x}{1 + x}\right)^{1/2} \quad , \tag{12.85}$$

and thus

$$(\Delta\phi)_1 = 4b^{-1} \quad , \tag{12.86}$$

leading to

$$\delta\phi = \frac{4m}{b} = \frac{4MG}{bc^2} \quad , \tag{12.87}$$

in agreement with the result (12.81) obtained above.

Derivation III: Linearising the Schwarzschild Metric

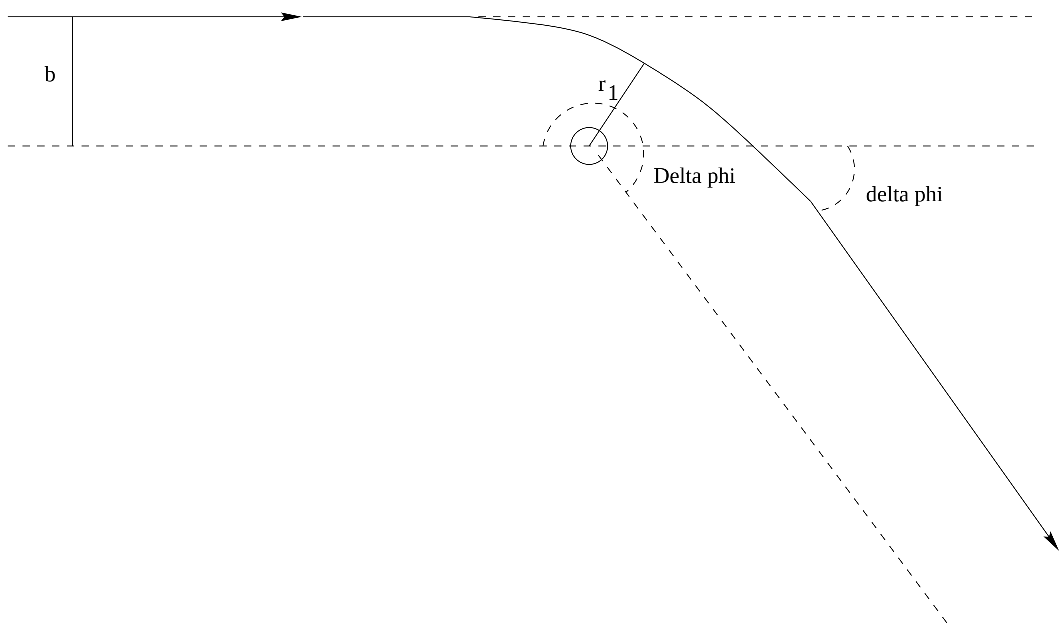

Figure 13: Bending of light by a star. Indicated are the definitions of the impact parameter b, the perihelion r_1, and of the angles $\Delta\phi$ and $\delta\phi$.

It is instructive to look at the second derivation from another point of view. As we will see, in some sense this derivation 'works' because the bending of light is accurately described by the *linearised solution*, i.e. by the metric that one obtains from the Schwarzschild metric by the approximation

$$\begin{aligned} A(r) = 1 - \frac{2m}{r} &\quad\rightarrow\quad 1 - \frac{2m}{r} \\ B(r) = (1 - \frac{2m}{r})^{-1} &\quad\rightarrow\quad 1 + \frac{2m}{r} \quad . \end{aligned} \tag{12.88}$$

I will only sketch the main steps in this calculation, so you should think of this subsection as an annotated exercise.

First of all, redoing the analysis of sections 12.1 and 12.2 for a general spherically symmetric static metric (11.5),

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad , \tag{12.89}$$

it is easy to see that the orbit equation can be written as

$$B(r)\frac{r'^2}{r^4} + \frac{1}{r^2} = \frac{\epsilon}{L^2} + \frac{E^2}{L^2 A(r)} \tag{12.90}$$

or, in terms of $u = 1/r$, as

$$Bu'^2 + u^2 = \frac{\epsilon}{L^2} + \frac{E^2}{L^2 A(u)} \quad . \tag{12.91}$$

We will concentrate on the lightlike case $\epsilon = 0$,

$$Bu'^2 + u^2 = \frac{E^2}{L^2 A(u)} \quad , \tag{12.92}$$

and express the impact parameter $b = L/E$ in terms of the turning point $r_1 = 1/u_1$ of the trajectory. At this turning point, $u' = 0$, and thus

$$\frac{E^2}{L^2} = A(u_1)u_1^2 \quad , \tag{12.93}$$

leading to

$$Bu'^2 = \frac{A(u_1)}{A(u)}u_1^2 - u^2 \quad . \tag{12.94}$$

We thus find

$$\frac{d\phi}{du} = \pm B(u)^{1/2}\left[\frac{A(u_1)}{A(u)}u_1^2 - u^2\right]^{-1/2} \quad . \tag{12.95}$$

For the (linearised) Schwarzschild metric the term in square brackets is

$$\begin{aligned}\frac{A(u_1)}{A(u)}u_1^2 - u^2 &= u_1^2(1 + 2m(u - u_1)) - u^2 \\ &= (u_1^2 - u^2)(1 - 2m\frac{u_1^2}{u_1 + u}) \quad .\end{aligned} \tag{12.96}$$

Using this and the approximate (linearised) value for $B(u)$,

$$B(u) \approx 1 + 2mu \tag{12.97}$$

one finds that $d\phi/du$ is given by

$$\begin{aligned}\frac{d\phi}{du} &= \pm B(u)^{1/2}\left[(u_1^2 - u^2)(1 - 2m\frac{u_1^2}{u_1 + u})\right]^{-1/2} \\ &\approx (u_1^2 - u^2)^{-1/2}\left(1 + m(\frac{u_1^2}{u_1 + u} + u)\right) \\ &= (u_1^2 - u^2)^{-1/2} + m\frac{u_1^3 - u^3}{(u_1^2 - u^2)^{3/2}} \quad .\end{aligned} \tag{12.98}$$

The first term now gives us the Newtonian result and, comparing with Derivation II, we see that the second term agrees precisely with the integrand of (12.84) with $b \to r_1$ (which, in a term that is already of order m, makes no difference). We thus conclude that the deflection angle is, as before,

$$\delta\phi = 2\int_0^{u_1} du\, m\frac{u_1^3 - u^3}{(u_1^2 - u^2)^{3/2}} = 4mu_1 \approx \frac{4m}{b} \quad . \tag{12.99}$$

This effect is physically measurable and was one of the first true tests of Einstein's new theory of gravity. For light just passing the sun the predicted value is

$$\delta\phi \sim 1,75'' \quad . \tag{12.100}$$

Experimentally this is a bit tricky to observe because one needs to look at light from distant stars passing close to the sun. Under ordinary circumstances this would not be observable, but in 1919 a test of this was performed during a total solar eclipse, by observing the effect of the sun on the apparent position of stars in the direction of the sun. The observed value was rather imprecise, yielding $1,5'' < \delta\phi < 2,2''$ which is, if not a confirmation of, at least consistent with General Relativity.

More recently, it has also been possible to measure the deflection of radio waves by the gravitational field of the sun. These measurements rely on the fact that a particular Quasar, known as 3C275, is obscured annually by the sun on October 8th, and the observed result (after correcting for diffraction effects by the corona of the sun) in this case is $\delta = 1,76'' \pm 0,02''$.

The value predicted by General Relativity is, interestingly enough, exactly twice the value that would have been predicted by the Newtonian approximation of the geodesic equation alone (but the Newtonian approximation is not valid anyway because it applies to slowly moving objects, and light certainly fails to satisfy this condition). A calculation leading to this wrong value had first been performed by Soldner in 1801 (!) (by cancelling the mass m out of the Newtonian equations of motion before setting $m = 0$) and also Einstein predicted this wrong result in 1908 (his equivalence principle days, long before he came close to discovering the field equations of General Relativity now carrying his name).

This result can be obtained from the above calculation by setting $B(u) = 1$ instead of (12.97), as in the Newtonian approximation only g_{00} is non-trivial. More generally, one can calculate the deflection angle for a metric with the approximate behaviour

$$B(u) \approx 1 + 2\gamma m u \quad , \tag{12.101}$$

for γ a real parameter, with the result

$$\delta\phi \approx \frac{1+\gamma}{2}\frac{4m}{b} \quad . \tag{12.102}$$

This reproduces the previous result for $\gamma = 1$, half its value for $\gamma = 0$, and checking to which extent measured deflection angles agree with the theoretical prediction of general relativity ($\gamma = 1$) constitutes an experimental test of general relativity. In this context γ is known as one of the PPN parameters (PPN for *parametrised post-Newtonian approximation*).

12.7 A Unified Description in terms of the Runge-Lenz Vector

The perhaps slickest way to obtain the orbits of the Kepler problem is to make use of the so-called Runge-Lenz vector. Recall that, due to conservation of angular momentum $\vec{L}$, the orbits in any spherically symmetric potential are planar. The bound orbits of the

Kepler problem, however, have the additional property that they are closed, i.e. that the perihelion is constant. This suggests that there is a further hidden symmetry in the Kepler problem, with the position of the perihelion the corresponding conserved charge. This is indeed the case.

Consider, for a spherically symmetric potential $W(r)$, the vector

$$\vec{A} = \dot{\vec{x}} \times \vec{L} + W(r)\vec{x} \tag{12.103}$$

or, in components,

$$A_i = \epsilon_{ijk}\dot{x}_j L_k + W(r)x_i \ . \tag{12.104}$$

A straightforward calculation, using the Newtonian equations of motion in the potential $W(r)$, shows that

$$\frac{d}{dt}A_i = (r\partial_r W(r) + W(r))\dot{x}_i \ . \tag{12.105}$$

Thus $\vec{A}$ is conserved if and only if $W(r)$ is homogeneous of degree (-1),

$$\frac{d}{dt}\vec{A} = 0 \quad \Leftrightarrow \quad W(r) = \frac{c}{r} \ . \tag{12.106}$$

In our notation, $c = \epsilon m$, and we will henceforth refer to the vector

$$\vec{A} = \dot{\vec{x}} \times \vec{L} + \frac{\epsilon m}{r}\vec{x} \tag{12.107}$$

as the *Runge-Lenz vector*.

It is well known, and easy to see by calculating e.g. the Poisson brackets of suitable linear combinations of $\vec{L}$ and $\vec{A}$, that $\vec{A}$ extends the manifest symmetry group of rotations $SO(3)$ of the Kepler problem to $SO(4)$ for bound orbits and $SO(3,1)$ for scattering orbits.

It is straightforward to determine the Keplerian orbits from $\vec{A}$. While $\vec{A}$ has 3 components, the only new information is contained in the direction of $\vec{A}$, as the norm A of $\vec{A}$ can be expressed in terms of the other conserved quantities and parameters (energy E, angular momentum L, mass m) of the problem. In the notation of section 12.1, one has

$$A^2 = E^2L^2 + \epsilon(L^2 + \epsilon m^2) \ . \tag{12.108}$$

Let us choose the constant direction $\vec{A}$ to be in the direction $\phi = 0$. Then $\vec{A}.\vec{x} = Ar\cos\phi$ and from (12.107) one finds

$$Ar\cos\phi = L^2 + \epsilon mr \ . \tag{12.109}$$

Now we consider the two cases $\epsilon = -1$ and $\epsilon = 0$.

For $\epsilon = -1$, (12.109) can be written as

$$\frac{1}{r(\phi)} = \frac{m}{L^2}(1 + \frac{A}{m}\cos\phi) \ . \tag{12.110}$$

Comparing with (12.43), we recognise this as the equation for an ellipse with eccentricity e and semi-major axis a (12.48) given by

$$e = \frac{A}{m} \qquad \frac{m}{L^2} = \frac{1}{a(1-e^2)} \quad . \tag{12.111}$$

Moreover, we see that the perihelion is at $\phi = 0$ which establishes that the Runge-Lenz vector points from the center of attraction to the (constant) position of the perihelion. During one revolution the angle ϕ changes from 0 to 2π.

For $\epsilon = 0$ (i.e. no potential), on the other hand, (12.109) reduces to

$$\frac{1}{r(\phi)} = \frac{A}{L^2}\cos\phi \tag{12.112}$$

This describes a straight line (12.73) with impact parameter

$$b = \frac{L^2}{A} = \frac{L}{E} \quad . \tag{12.113}$$

In this case, ϕ runs from $-\pi/2$ to $\pi/2$ and the point of closest approach is again at $\phi = 0$ (distance b).

We see that the Runge-Lenz vector captures precisely the information that in the Newtonian theory bound orbits are closed and light-rays are not deflected. The Runge-Lenz vector will no longer be conserved in the presence of the general relativistic correction to the Newtonian motion, and this non-constancy is a precise measure of the deviation of the general relativistic orbits from their Newtonian counterparts. As shown e.g. in an article by Brill and Goel[3] this provides a very elegant and quick way of (re-)deriving the results about perihelion precession and deflection of light in the solar system.

Calculating the time-derivative of $\vec{A}$ (12.107) for a particle moving in the general relativistic potential (12.16)

$$V(r) = \epsilon\frac{m}{r} - \frac{mL^2}{r^3} \quad , \tag{12.114}$$

one finds (of course we now switch from t to τ)

$$\frac{d}{d\tau}\vec{A} = \frac{3m^2L^2}{r^2}\frac{d}{d\tau}\vec{n} \tag{12.115}$$

where $\vec{n} = \vec{x}/r = (\cos\phi, \sin\phi, 0)$ is the unit vector in the plane $\theta = \pi/2$ of the orbit. Thus $\vec{A}$ rotates with angular velocity

$$\omega = \frac{3mL^2\cos\phi}{Ar^2}\dot{\phi} \quad . \tag{12.116}$$

Here A now refers to the norm of the Newtonian Runge-Lenz vector (12.107) calculated for a trajectory $\vec{x}(\tau)$ in the general relativistic potential (12.114). This norm is now no longer constant,

$$A^2 = E^2L^2 + \epsilon(L^2 + \epsilon m^2) + \frac{2mL^4}{r^3} \quad . \tag{12.117}$$

[3] D. Brill, D. Goel, *Light bending and perihelion precession: A unified approach*, Am. J. Phys. 67 (1999) 316, `arXiv:gr-qc/9712082`

However, assuming that the change in $\vec{A}$ is small, we obtain an approximate expression for ω by substituting the unperturbed orbit $r_0(\phi)$ from (12.109),

$$r_0(\phi) = \frac{L^2}{A\cos\phi - \epsilon m} \quad , \tag{12.118}$$

as well as the unperturbed norm (12.108) in (12.116) to find

$$\omega \approx \frac{3m}{AL^2}(A\cos\phi - \epsilon m)^2 \cos\phi \, \dot{\phi} \quad . \tag{12.119}$$

Now the total change in the direction of $\vec{A}$ when the object moves from ϕ_1 to ϕ_2 can be calculated from

$$\begin{aligned} \delta\phi &= \int_{\phi_1}^{\phi_2} \omega d\tau \\ &= \frac{3m}{AL^2}\int_{\phi_1}^{\phi_2} d\phi \, (A\cos\phi - \epsilon m)^2 \cos\phi \quad . \end{aligned} \tag{12.120}$$

For $\epsilon = -1$, and $(\phi_1, \phi_2) = (0, 2\pi)$, this results in (only the $\cos^2\phi$-term gives a non-zero contribution)

$$\delta\phi = 2\pi\frac{3m^2}{L^2} = \frac{6\pi m^2}{L^2} \quad , \tag{12.121}$$

in precise agreement with (12.58,12.60).

For $\epsilon = 0$, on the other hand, one has

$$\delta\phi = \frac{3mA}{L^2}\int_{-\pi/2}^{\pi/2} d\phi \, \cos^3\phi \quad . \tag{12.122}$$

Using

$$\int \cos^3\phi = \sin\phi - \tfrac{1}{3}\sin^3\phi \quad , \tag{12.123}$$

one finds

$$\delta\phi = \frac{4mA}{L^2} = \frac{4m}{b} \quad , \tag{12.124}$$

which agrees precisely with the results of section 12.6.

13 Black Holes: Approaching and Crossing the Schwarzschild Radius

So far, we have been considering objects of a size larger (in practice much larger) than their Schwarzschild radius, $r_0 > r_S$. We also noted that the effective potential $V_{eff}(r)$ is perfectly well behaved at r_S. We now consider objects with $r_0 < r_S$ and try to unravel some of the bizarre physics that nevertheless occurs when one approaches or crosses $r_S = 2m$. We will do this in several steps:

- First we will consider observers that don't quite dare to cross r_S and which try to remain stationary at a fixed value of r close to r_S.
- Then we will consider observers that fall freely (and radially) in this geometry, and describe their voyage both from their point of view and from that of a distant stationary observer (which will turn out to be quite different).
- Then we will study the geometry of the Schwarzschild metric near $r = r_S$, and show that the geoemtry is completely non-singular (and is in fact closely related to the geometry of the Rindler metric for Minkowski space-time discussed in section 1.2).
- These considerations will indicate that (and explain why) the usual Schwarzschild coordinates (specifically the time-coordinate t) are inadequate for describing the physics across the radius r_S.
- In order to learn more about the region near and beyond $r = r_S$, we next study the behaviour of lightcones and light rays in this geometry.
- These considerations will then lead us to the introduction of coordinates in which the Schwarzschild metric is non-singular for all $0 < r < \infty$, and then the fun can begin and we can try to understand what actually happens (and what characterises) $r = r_S$.

13.1 Stationary Observers

Some insight into the Schwarzschild geometry, and the difference between Newtonian gravity and general relativistic gravity, is provided by looking at stationary observers, i.e. observers hovering at fixed values of (r, θ, ϕ). Thus their 4-velocity $u^\alpha = \dot{x}^\alpha$ has the form $u^\alpha = (u^0, 0, 0, 0)$ with $u^0 > 0$. The normalisation $u^\alpha u_\alpha = -1$ then implies

$$u^\alpha = \left(\frac{1}{\sqrt{1 - 2m/r}}, 0, 0, 0 \right) \quad . \tag{13.1}$$

The worldline of a stationary observer is clearly not a geodesic (that would be the worldline of an observer freely falling in the gravitational field), and we can calculate its covariant acceleration (4.61)

$$a^\alpha = \frac{D}{D\tau}u^\alpha = u^\beta \nabla_\beta u^\alpha = (d/d\tau)u^\alpha + \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma \ . \tag{13.2}$$

Noting that u^α is time-independent, one finds

$$a^\alpha = \Gamma^\alpha_{00}(u^0)^2 = \Gamma^\alpha_{00}(1 - 2m/r)^{-1} \ . \tag{13.3}$$

Thus only a^r is non-zero, and one finds

$$a^\alpha = (0, m/r^2, 0, 0) \ . \tag{13.4}$$

This looks nicely Newtonian, with a force in the radial direction designed to precisely cancel the gravitational attraction. However, this is a bit misleading since this is a coordinate dependent statement. A coordinate-invariant quantity is the norm of the acceleration,

$$\mathsf{a}(r) \equiv \left(g_{\alpha\beta}a^\alpha a^\beta\right)^{1/2} = \frac{m}{r^2}\left(1 - \frac{2m}{r}\right)^{-1/2} \ . \tag{13.5}$$

(One can also think of this as the radial component of the acceleration with respect to an orthonormal frame at that point, whose radial component would be $(1-2m/r)^{1/2}\partial_r$, the prefactor ensuring that this vector has unit length.) While this approaches the Newtonian value as $r \to \infty$, it diverges as $r \to 2m$, indicating that stationary observers will find it harder and harder, and need to travel nearly at the speed of light, to remain stationary close to $r = 2m$.

13.2 Vertical Free Fall

We will now consider an object with $r_0 < r_s$ and an observer who is freely falling vertically (radially) towards such an object. "Vertical" means that $\dot\phi = 0$, and therefore there is no angular momentum, $L = 0$. Hence the effective potential equation (12.13) becomes

$$E^2 - 1 = \dot r^2 - \frac{2m}{r} \ . \tag{13.6}$$

In particular, if r_i is the point at which the particle (observer) A was initially at rest,

$$\frac{dr}{d\tau}|_{r=r_i} = 0 \ , \tag{13.7}$$

we have the relation

$$E^2 = 1 - \frac{2m}{r_i} \tag{13.8}$$

between the constant of motion E and the initial condition r_i. In particular, $E = 1$ for an observer initially at rest at infinity. Then we obtain

$$\dot r^2 = \frac{2m}{r} - \frac{2m}{r_i} \tag{13.9}$$

and, upon differentiation,

$$\ddot{r} + \frac{m}{r^2} = 0 \ . \tag{13.10}$$

This is just like the Newtonian equation (which should not come as a surprise as V_{eff} coincides with the Newtonian potential for zero angular momentum $L = 0$), apart from the fact that r is *not* radial distance and the familiar $\tau \neq t$. Nevertheless, calculation of the time τ along the path proceeds exactly as in the Newtonian theory. For the proper time required to reach the point with coordinate value $r = r_1$ we obtain

$$\tau = -(2m)^{-1/2} \int_{r_i}^{r_1} dr \, \left(\frac{r_i r}{r_i - r} \right)^{1/2} \ . \tag{13.11}$$

Since this is just the Newtonian integral, we know, even without calculating it, that it is finite as $r_1 \to r_S$ and even as $r_1 \to 0$. This integral can also be calculated in closed form, e.g. via the change of variables

$$\frac{r}{r_i} = \sin^2 \alpha \qquad \alpha_1 \leq \alpha \leq \frac{\pi}{2} \ , \tag{13.12}$$

leading to

$$\tau = 2 \left(\frac{r_i^3}{2m} \right)^{1/2} \int_{\alpha_1}^{\pi/2} d\alpha \, \sin^2 \alpha = \left(\frac{r_i^3}{2m} \right)^{1/2} \left[\alpha - \tfrac{1}{2} \sin 2\alpha \right]_{\alpha_1}^{\pi/2} \ . \tag{13.13}$$

In particular, this is finite as $r_1 \to 2m$ and our freely falling observer can reach and cross the Schwarzschild radius r_S in finite proper time.

Coordinate time, on the other hand, becomes infinite at $r_1 = 2m$. This can roughly (and very easily) be seen by noting that

$$\Delta\tau = (1 - \frac{2m}{r})^{1/2} \Delta t \ . \tag{13.14}$$

As $\Delta\tau$ is finite (as we have seen) and $(1 - \frac{2m}{r})^{1/2} \to 0$ as $r \to 2m$, clearly we need $\Delta t \to \infty$. We will now describe this in a more precise and quantitative way.

13.3 Vertical Free Fall as seen by a Distant Observer

We will now investigate how the above situation presents itself to a distant observer hovering at a fixed radial distance r_∞. He will observe the trajectory of the freely falling observer as a function of his proper time τ_∞. Up to a constant factor $(1 - 2m/r_\infty)^{1/2}$, this is the same as coordinate time t, and we will lose nothing by expressing r as a function of t rather than as a function of τ_∞.

From (13.6),

$$\dot{r}^2 + (1 - \frac{2m}{r}) = E^2 \ , \tag{13.15}$$

which expresses r as a function of the freely falling observer's proper time τ, and the definition of E,

$$E = \dot{t}(1 - \frac{2m}{r}) \ , \tag{13.16}$$

which relates τ to the coordinate time t, one finds an equation for r as a function of t,

$$\frac{dr}{dt} = -E^{-1}(1 - \frac{2m}{r})(E^2 - (1 - \frac{2m}{r}))^{1/2} \tag{13.17}$$

(the minus sign has been chosen because r decreases as t increases). We want to analyse the behaviour of the solution of this equation as the freely falling observer approaches the Schwarzschild radius, $r \to 2m$,

$$\begin{aligned} \frac{dr}{dt} &= -E^{-1}(\frac{r-2m}{r})(E^2 - \frac{r-2m}{r})^{1/2} \\ &\to -E^{-1}(\frac{r-2m}{2m})(E^2)^{1/2} = -(\frac{r-2m}{2m}) \ . \end{aligned} \tag{13.18}$$

We can write this equation as

$$\frac{d}{dt}(r - 2m) = -\frac{1}{2m}(r - 2m) \ , \tag{13.19}$$

which obviously has the solution

$$(r - 2m)(t) \propto \mathrm{e}^{-t/2m} \ . \tag{13.20}$$

This shows that, from the point of view of the observer at infinity, the freely falling observer reaches $r = 2m$ only as $t \to \infty$. In particular, the distant observer will never actually see the infalling observer cross the Schwarzschild radius.

This is clearly an indication that there is something wrong with the time coordinate t which runs too fast as one approaches the Schwarzschild radius. We can also see this by looking at the coordinate velocity $v = dr/dt$ as a function of r. Let us choose $r_i = \infty$ for simplicity - other choices will not change our conclusions as we are interested in the behaviour of $v(r)$ as $r \to r_S$. Then $E^2 = 1$ and from (13.17) we find

$$v(r) = -(2m)^{1/2}\frac{r - 2m}{r^{3/2}} \tag{13.21}$$

As a function of r, $v(r)$ reaches a maximum at $r = 6m = 3r_S$, where the velocity is (restoring the velocity of light c)

$$v_{max} = v(r = 6m) = \frac{2c}{3\sqrt{3}} \ . \tag{13.22}$$

Beyond that point, $v(r)$ decreases again and clearly goes to zero as $r \to 2m$. The fact that the coordinate velocity goes to zero is another manifestation of the fact that coordinate time goes to infinity. Somehow, the Schwarzschild coordinates are not suitable for describing the physics at or beyond the Schwarzschild radius because the time coordinate one has chosen is running too fast. This is the crucial insight that will allow us to construct 'better' coordinates, which are also valid for $r < r_S$, later on in this section.

13.4 Infinite Gravitational Red-Shift

One dramatic aspect of what is happening at (or, better, near) the Schwarzschild radius for very (very!) compact objects with $r_S > r_0$ is the following. Recall the formula (2.86) for the gravitational red-shift, which gave us the ratio between the frequency of light ν_e emitted at the radius r_e and the frequency ν_∞ received at the radius $r_\infty > r_e$ in a static spherically symmetric gravitational field. The result, which is in particular also valid for the Schwarzschild metric, was

$$\frac{\nu_\infty}{\nu_e} = \frac{(-g_{00}(r_e))^{1/2}}{(-g_{00}(r_\infty))^{1/2}} \quad . \tag{13.23}$$

In the case of the Schwarzschild metric, this is

$$\frac{\nu_\infty}{\nu_e} = \frac{(1 - 2m/r_e)^{1/2}}{(1 - 2m/r_\infty)^{1/2}} \quad . \tag{13.24}$$

We now choose the emitter to be the freely falling observer whose position is described by $r_e = r(\tau)$ or $r(t)$, and the receiver to be the fixed observer at $r_\infty \gg r_S$. As $r_e \to r_S$, one clearly finds

$$\frac{\nu_\infty}{\nu_e} \to 0 \quad . \tag{13.25}$$

Expressed in terms of the gravitational red-shift factor z,

$$1 + z = \frac{\nu_e}{\nu_\infty} \tag{13.26}$$

this means that there is an *infinite gravitational red-shift* as $r_e \to r_S$,

$$r_e \to r_S \quad \Rightarrow \quad z \to \infty \quad . \tag{13.27}$$

More explicitly, using (13.20) one finds the late-time behaviour of the red-shift factor z, in terms of coordinate time (or the distant observer's proper time), to be

$$1 + z \propto (r - 2m)(t)^{-1/2} \propto \mathrm{e}^{t/4m} \quad . \tag{13.28}$$

Thus for the distant observer at late times there is an exponentially growing red-shift and the distant observer will never actually see the unfortunate emitter crossing the Schwarzschild radius: he will see the freely falling observer's signals becoming dimmer and dimmer and arriving at greater and greater intervals, and the freely falling observer will completely disappear from the distant observer's sight as $r_e \to r_S$. Note that the time-scale t_z for this exponential red-shift at late times is set by $t_z \sim 4m/c$, which is of the order of

$$t_z \sim 10^{-5}\mathsf{s}\left(\frac{M}{M_{\mathsf{sun}}}\right) \quad , \tag{13.29}$$

so that this is pretty much instantaneous for an object the mass of an ordinary star. We will come back to these estimates later on when (briefly) talking about gravitational collapse in section 13.9.

This was for a stationary observer. As we have seen, the situation presents itself rather differently for the freely falling observer himself who will not immediately notice anything particularly dramatic happening as he approaches or crosses r_S.

13.5 The Geometry Near r_S and Minkowski Space in Rindler Coordinates

We have now seen in two different ways why the Schwarzschild coordinates are not suitable for exploring the physics in the region $r \leq 2m$: in these coordinates the metric becomes singular at $r = 2m$ and the coordinate time becomes infinite. On the other hand, we have seen no indication that the local physics, expressed in terms of covariant quantities like proper time or the geodesic equation, becomes singular as well. So we have good reasons to suspect that the singular behaviour we have found is really just an artefact of a bad choice of coordinates.

In fact, the situation regarding the Schwarzschild coordinates is quite reminiscent of the Rindler coordinates for Minkowski space we discussed (way back) in section 1.2. As we saw there, these only covered part of Minkowski space (the right quadrant or *Rindler wedge*), bounded by lines (or hypersurfaces) where the time coordinate η became infinite, while inertial (geodesic, freely falling) observers could exit this region in finite proper time. This is in fact more than just a loose analogy: as we will see now, the Rindler metric (1.24) gives an accurate description of the geometry of the Schwarzschild metric close to the Schwarzschild radius.

To confirm this, let us temporarily introduce the variable $\tilde{r} = r - 2m$ measuring the coordinate distance from the horizon. In term of $\tilde{r}$ the (t, r)-part of the Schwarzschild metric reads

$$ds^2 = -\left(\frac{\tilde{r}}{\tilde{r}+2m}\right)dt^2 + \left(\frac{\tilde{r}+2m}{\tilde{r}}\right)d\tilde{r}^2 \quad . \tag{13.30}$$

Close to the horizon, i.e. for small $\tilde{r} << 2m$, we can approximate

$$\frac{\tilde{r}}{\tilde{r}+2m} \approx \frac{\tilde{r}}{2m} \qquad \frac{\tilde{r}+2m}{\tilde{r}} \approx \frac{2m}{\tilde{r}} \quad , \tag{13.31}$$

so that the metric becomes

$$ds^2 = -\frac{\tilde{r}}{2m}dt^2 + \frac{2m}{\tilde{r}}d\tilde{r}^2 \quad . \tag{13.32}$$

Introducing the new radial variable ρ (proper radial distance from the horizon) via

$$d\rho^2 = \frac{2m}{\tilde{r}}d\tilde{r}^2 \quad \Rightarrow \quad \rho = \sqrt{8m\tilde{r}} \quad , \tag{13.33}$$

one finds

$$ds^2 = -\frac{1}{16m^2}\rho^2 dt^2 + d\rho^2 \quad . \tag{13.34}$$

Finally a simple rescaling of t, $\eta = t/4m$, leads to

$$ds^2 = -\rho^2 d\eta^2 + d\rho^2 \quad , \tag{13.35}$$

which, remarkably, is identical to the Rindler metric (1.24). Keeping track of the transverse 2-sphere and using $r^2 \approx (2m)^2$ in the near-horizon approximation, the complete metric in this limit and in these coordinates reads

$$ds^2 = -\rho^2 d\eta^2 + d\rho^2 + (2m)^2 d\Omega^2 \tag{13.36}$$

If we further restrict to just a small angular region on the sphere, we can approximate

$$d\theta^2 + \sin^2\theta d\phi^2 \approx d\theta^2 + \theta^2 d\phi^2 = (dx^2)^2 + (dx^3)^2 \tag{13.37}$$

which gives us the complete 4-dimensional Rindler metric.

In any case, we see that, up to the harmless redefinition $(r, t) \to (\rho, \eta)$ that we just performed, Schwarzschild coordinates for the Schwarzschild geometry are just like Rindler coordinates for Minkowski space.

The first, and most crucial, thing we learn from this is that the singularity of the Schwarzschild metric at $r = 2m$ in the Schwarzschild coordinates (t, r) is, as anticipated, a mere coordinate singularity. Indeed, $r = 2m$ corresponds to $\tilde{r} = 0 \Leftrightarrow \rho = 0$, and we already know that the singularity of the Rindler metric at $\rho = 0$ is just a coordinate singularity (which can be eliminated e.g. by passing to standard inertioal Minkowski coordinates ξ^A via (1.23)).

Moreover, we can now understand physically why the Schwarzschild coordinates break down at $r = 2m$: they are adapted to accelerating observers, in the present context the stationary observers in the Schwarzschild geometry at constant r, with proper time proportional to the Schwarzschild coordinate time t. Referring back to Figure 7 in section 1.2, these are the observers with constant ρ hyperbolic wold lines.

The problem is evidently that the required acceleration of these observers becomes infinite as $\rho \to 0 \Leftrightarrow r \to 2m$ (as we have calculated in section 13.1). That these observers appear to see a singular metric is then not the geometry's fault but can be attributed to a bad choice of observers whose perception of the geometry is distorted by their acceleration and their desperate attempt to stay at constant values of r even when they are very close to $r = 2m$.

The situation is quite different for the freely falling observers. Their worldlines look like the vertical line labelled "worldline of a stationary observer" in Figure 7, they cross the horizon in finite proper time, experiencing no strong acceleration or gravitational fields. As already noted in section 1.2, they evidently become invisible to observers in the Rindler quadrant (now "Schwarzschild patch") of the geometry, stationary outside observers noting an infnite gravitational redshift affecting the signals sent out by the freely falling observer. Introducing coordinates adapted to them (so that e.g. time is their proper time) would be tantamount to passing from Rindler coordinates to ordinary Minkowski coordinates.

We will not quite do that (one could) but base our construction of new coordinates not on the behaviour of timelike freely falling observers but ingoing light rays which evidently also do not suffer from the problems of the stationary observers (in the Rindler/Minkowski case this amounts to the same thing in the end).

Finally, we can anticipate that upon introduction of suitable analogues of the Minkowski coordinates for the Rindler space-time, we may perhaps uncover not just one new region (quadrant) of space-time (the one lying to the "future" of $r = 2m$), but also counterparts of the other two quadrants of Minkowski space. This expectation will indeed be borne out.

13.6 Tortoise Coordinates

To improve our understanding of the Schwarzschild geometry, it is important to study its *causal structure*, i.e. the lightcones. Radial null curves satisfy

$$(1 - 2m/r)dt^2 = (1 - 2m/r)^{-1}dr^2 \quad . \tag{13.38}$$

Thus

$$\frac{dt}{dr} = \pm(1 - 2m/r)^{-1} \quad , \tag{13.39}$$

In the (r, t)-diagram of Figure 14, dt/dr represents the slope of the lightcones at a given value of r. Now, as $r \to 2m$, one has

$$\frac{dt}{dr} \stackrel{r\to 2m}{\longrightarrow} \pm\infty \quad , \tag{13.40}$$

so the light clones 'close up' as one approaches the Schwarzschild radius. This is the same statement as before regarding the fact that the coordinate velocity goes to zero at $r = 2m$, but this time for null rather than timelike geodesics.

As our first step towards introducing coordinates that are more suitable for describing the region around r_S, let us write the Schwarzschild metric in the form

$$ds^2 = (1 - 2m/r)(-dt^2 + (1 - 2m/r)^{-2}dr^2) + r^2d\Omega^2 \quad . \tag{13.41}$$

We see that it is convenient to introduce a new radial coordinate r^* via

$$dr^* = (1 - 2m/r)^{-1}dr \quad . \tag{13.42}$$

The solution to this equation is

$$r^* = r + 2m\log(r/2m - 1) \quad . \tag{13.43}$$

This new radial coordinate r^*, known as the *tortoise coordinate*, also provides us with the solution

$$t = \pm r^* + C \tag{13.44}$$

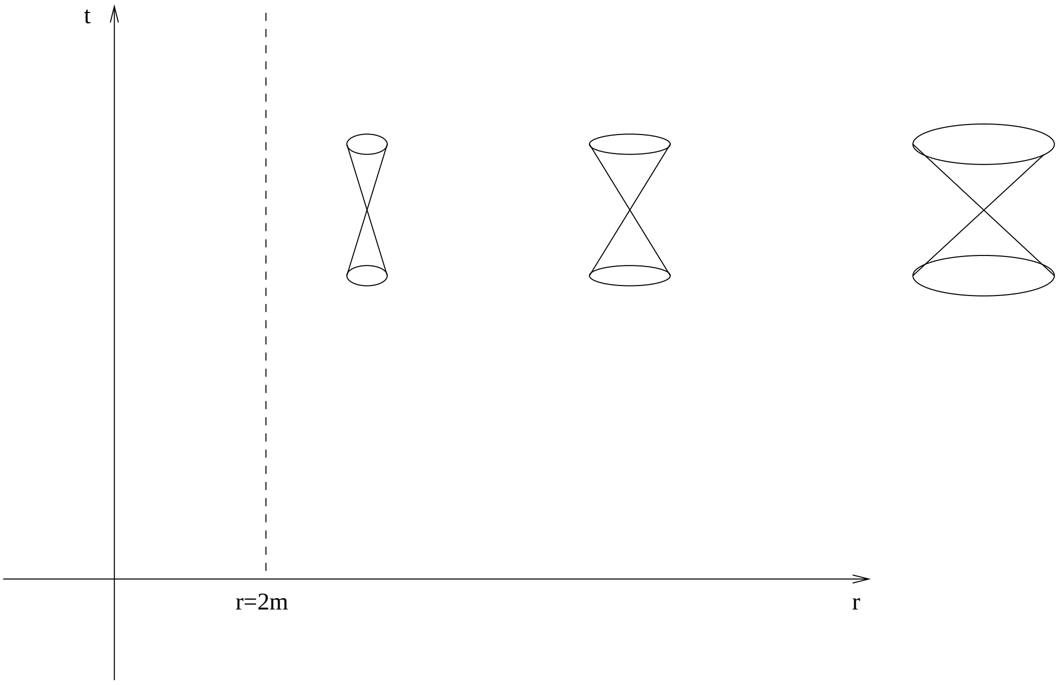

Figure 14: The causal structure of the Schwarzschild geometry in the Schwarzschild coordinates (r, t). As one approaches $r = 2m$, the lightcones become narrower and narrower and eventually fold up completely.

to the equation (13.39) describing the lightcones. In terms of r^* the metric simply reads

$$ds^2 = (1 - 2m/r)(-dt^2 + dr^{*2}) + r^2 d\Omega^2 \quad , \tag{13.45}$$

where r is to be thought of as a function of r^*.

We see immediately that we have made some progress. Now the lightcones, defined by

$$dt^2 = dr^{*2} \quad , \tag{13.46}$$

do not seem to fold up as the lightcones have the constant slope $dt/dr^* = \pm 1$ (see Figure 15), and there is no singularity at $r = 2m$. However, r^* is still only defined for $r > 2m$ and the surface $r = 2m$ has been pushed infinitely far away ($r = 2m$ is now at $r^* = -\infty$). Moreover, even though non-singular, the metric components g_{tt} and $g_{r^*r^*}$ (as well as $\sqrt{g}$) vanish at $r = 2m$.

13.7 Eddington-Finkelstein Coordinates, Black Holes and Event Horizons

Part of the problem is that t is still one of our coordinates, while we had already anticipated that t is not suitable for exploring the region beyond r_S. On the other hand, geodesics reach r_S in finite proper or affine time. It is therefore natural to introduce

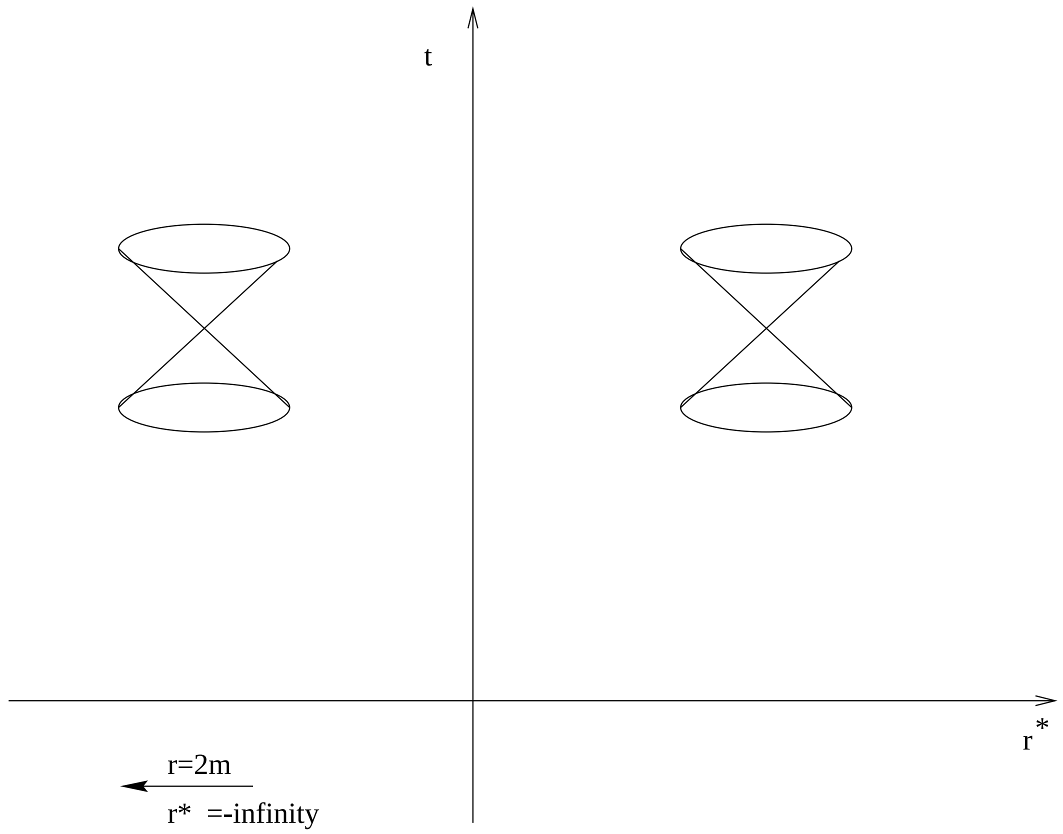

Figure 15: The causal structure of the Schwarzschild geometry in the tortoise coordinates (r^*, t). The lightcones look like the lightcones in Minkowski space and no longer fold up as $r \to 2m$ (which now sits at $r^* = -\infty$).

coordinates that are adapted to null geodesics, by promoting the integration constant C in (13.44) to a new coordinate, namely the retarded and advanced time coordinates

$$u = t - r^* \quad , \quad v = t + r^* \ . \tag{13.47}$$

Then infalling radial null geodesics ($dr^*/dt = -1$) are characterised by $v = const.$ and outgoing radial null geodesics by $u = const.$ [It is also possible, and occasionally convenient, to introduce coordinates adapted to timelike geodesics.]

Now we pass to the ingoing or outgoing *Eddington-Finkelstein* coordinates (v, r, θ, ϕ) or (u, r, θ, ϕ) (note that we keep r but eliminate t). This coordinate transformation $v(t, r) = t + r^*$ is of the general form $T(t, r) = t + \psi(r)$ (11.2) disucussed previously, and preserves the t-independence and manifest spherical symmetry of the metric. In terms of these coordinates the Schwarzschild metric reads

$$\begin{aligned} ds^2 &= -(1 - 2m/r)dv^2 + 2dv\, dr + r^2 d\Omega^2 \\ &= -(1 - 2m/r)du^2 - 2du\, dr + r^2 d\Omega^2 \ . \end{aligned} \tag{13.48}$$

Even though the metric coefficent g_{uu} or g_{vv} vanishes at $r = 2m$, there is no real degeneracy, the two-dimensional metric in the (v, r)- or (u, r)-directions having the completely non-singular and non-degenerate form

$$g = \begin{pmatrix} -(1 - 2m/r) & \pm 1 \\ \pm 1 & 0 \end{pmatrix} \tag{13.49}$$

In particular, the determinant of the metric is

$$-\det(g_{\mu\nu}) \equiv g = r^4 \sin^2\theta \ , \tag{13.50}$$

which is completely regular at $r = 2m$.

To determine the lightcones in the Eddington-Finkelstein coordinates we again look at radial null geodesics which this time are solutions to

$$(1 - 2m/r)dv^2 = 2dv\, dr \ . \tag{13.51}$$

Thus either $dv/dr = 0$ which, as we have seen, describes incoming null geodesics, $v = const.$, or

$$\frac{dv}{dr} = 2(1 - 2m/r)^{-1} \ , \tag{13.52}$$

which then describes outgoing null geodesics (the solution to this equation is evidently $v = 2r^* + C \Leftrightarrow u = t - r^* = C$). Thus the lightcone remain well-behaved (do not fold up) at $r = 2m$, the surface $r = 2m$ is at a finite coordinate distance, namely (to reiterate the obvious) at $r = 2m$, and there is no problem with following geodesics beyond $r = 2m$.

But even though the lightcones do not fold up at $r = 2m$, something interesting is certainly hapening there. Whereas, in a (v, r)-diagram (see Figure 16), one side of the

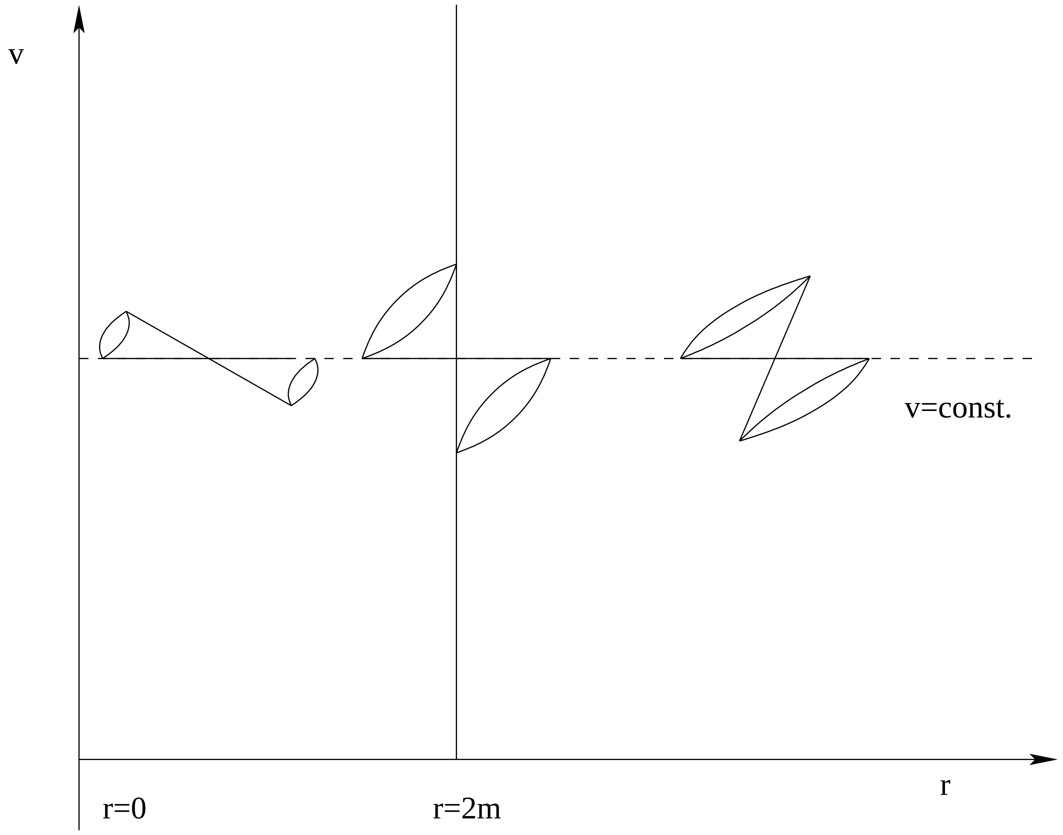

Figure 16: The behaviour of lightcones in ingoing Eddington-Finkelstein coordinates. Lightcones do not fold up at $r = 2m$ but tilt over so that for $r < 2m$ only movement in the direction of decreasing r towards the singularity at $r = 0$ is allowed.

lightcone always remains horizontal (at $v = const.$), the other side becomes vertical at $r = 2m$ ($dv/dr = \infty$) and then tilts over to the other side. In particular, beyond $r = 2m$ all future-directed paths, those within the forward lightcone, now have to move in the direction of decreasing r: clearly the ingoing null geodesics move towards smaller values of r, but so do those that for $r > 2m$ were outgoing,

$$(13.52) \quad \Rightarrow \quad dr/dv < 0 \quad \text{for} \quad r < 2m \ . \tag{13.53}$$

There is thus no way to turn back to larger values of r, not on a geodesic but also not on any other path (i.e. not even with a powerful rocket) once one has gone past $r = 2m$.

Thus, even though locally the physics at $r = 2m$ is well behaved, globally the surface $r = 2m$ is very significant as it is a point of no return. Once one has passed the Schwarzschild radius, there is no turning back. Such a surface is known as an *event horizon*. Note that this is a null surface so, in particular, once one has reached the event horizon one has to travel at the speed of light to stay there and not be forced further towards $r = 0$.

In any case, we now encounter no difficulties when entering the region $r < 2m$, e.g. along lines of constant u and this region should be included as part of the physical space-time. Note that because $v = t + r^*$ and $r^* \to -\infty$ for $r \to 2m$, we see that decreasing r along lines of constant v amounts to $t \to \infty$. Thus the new region at $r \leq 2m$ we have discovered is in some sense a future extension of the original Schwarzschild space-time.

Note also that nothing, absolutely nothing, no information, no light ray, no particle, can escape from the region behind the horizon. Thus we have a *Black Hole*, an object that is (classically) completely invisible. Even though the Eddington-Finkelstein coordinates were already introduced by Eddington back in 1924, the full significance of the Schwarzschild radius and its interepretation as a "one-way membrane" were only understood much later (Finkelstein, 1958).

In the above (v, r) coordinate system we can cross the event horizon only on future directed paths, not on past directed ones. The situation is reversed when one uses the coordinates (u, r) instead of (v, r). In that case, the lightcones in Figure 16 are flipped (either up-down or left-right), and one can pass through the horizon on past directed curves. The new region of space-time covered by the coordinates (u, r) is definitely different from the new region we uncovered using (v, r) even though both of them lie 'behind' $r = 2m$. In fact, this one is a past extension (beyond $t = -\infty$) of the original Schwarzschild 'patch' of space-time. In this patch, the region behind $r = 2m$ acts like the opposite (time-reversal) of a black hole (a white hole) which cannot be entered on any future-directed path.

Ordinary space-time diagrams are more familiar (and therefore more intuitive) than space-null diagrams such as the above (r, v)-diagram (in which, for example, in the asymptoticaly flat regime $r \to \infty$ the lightcone has slopes 0 and 2 rather than the usual $\pm 45°$ slopes ± 1). In the present case this can easily be rectified by introducing a new time-coordinate $\tilde{t}$ instead of v by the relation

$$v = t + r^* = \tilde{t} + r \quad . \tag{13.54}$$

The metric looks perhaps somewhat less illuminating in the $(\tilde{t}, r)$-coordinates,

$$ds^2 = -(1 - 2m/r)d\tilde{t}^2 + (4m/r)d\tilde{t}\, dr + (1 + 2m/r)dr^2 + r^2 d\Omega^2 \quad , \tag{13.55}$$

but the lightcones and the horizon now have the following simple and easy to visualise description:

- one side of the lightcone has (as we already know) $v = const.$, thus $d\tilde{t}/dr = -1$ everywhere, is thus at $-45°$ everywhere.
- the other side of the lightcone is described by $d\tilde{t}/dr = +1$ for $r \to \infty$ (slope $+1$), by $d\tilde{t}/dr \to \infty$ for $r \to 2m$, and $d\tilde{t}/dr \to -1$ for $r \to 0$.
- In particular, the horizon is (again) vertical in such a diagram, with the (would-be) outgoing side of the lightcone tangent to the horizon, while the lightcone degenerates as $r \to 0$.

Nice diagrams that you can find in many places depicting the collapse of a spherically symmetric star to a black hole and the formation of the horizon typically (either implicitly or explicitly) use these $(\tilde{t}, r)$-coordinates.

13.8 The Klein-Gordon Field in the Schwarzschild Geometry

As an aside, I just want to point out that the tortoise coordinate r^* and the Eddington Finkelstein retarded and advanced time coordinates u and v are not only useful for clarifying the causal structure of the Schwarzschild geometry, but also for the analysis of the propagation of scalar (and other) fields. This is principally due to the fact that in these coordinates the (t, r)-part of the metric is conformally flat - see (13.45) or (13.66) below. Combined with the observation of section 5.5 that the Klein-Gordon action is conformally invariant in $(1+1)$-dimensions, this leads to a canonical form for the $(3+1)$ wave operator in the (t, r)-sector (and the (θ, ϕ)-sector is standard anyway).

Specifically, we start with the action for a (massless, say) scalar field ϕ in the metric (13.45),

$$\begin{aligned} S[\phi] &\sim \int \sqrt{g} d^4x \; g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \\ &= \int dt \; dr^* \; d\Omega \; r^2 f(r) \left[f(r)^{-1} (-(\partial_t \phi)^2 + (\partial_{r^*} \phi)^2) - r^{-2} \phi \Delta_{S^2} \phi \right] \\ &= \int dt \; dr^* \; d\Omega \; \left[-(r \partial_t \phi)^2 + (r \partial_{r^*} \phi)^2) - f(r) \phi \Delta_{S^2} \phi \right] \quad , \end{aligned} \tag{13.56}$$

where $f(r) = 1 - 2m/r$, $d\Omega = \sin\theta d\theta \; d\phi$ denotes the solid angle on the 2-sphere, and Δ_{S^2} is the Laplace operator on the 2-sphere. Separating variables according to

$$\phi(x) = r^{-1} \sum_{\ell,m} \psi_{\ell m}(t, r^*) Y_{\ell m}(\theta, \phi) \quad , \tag{13.57}$$

using

$$\Delta_{S^2} Y_{\ell m} = -\ell(\ell+1) Y_{\ell m} \quad , \tag{13.58}$$

and using

$$\partial_{r^*} r = f(r) \quad \Rightarrow \quad r \partial_{r^*} (\psi_{\ell m}/r) = \partial_{r^*} \psi_{\ell m} - r^{-1} f(r) \psi_{\ell m} \quad , \tag{13.59}$$

one finds for $\psi = \psi_{\ell m}$ the equation of motion

$$(\partial_t^2 - \partial_{r^*}^2) \psi + V_\ell(r^*) \psi = 0 \quad , \tag{13.60}$$

where

$$V_\ell(r^*) = f(r) \left(\frac{\ell(\ell+1)}{r^2} + \frac{2m}{r^3} \right) \quad . \tag{13.61}$$

This potential is non-negative for all $2m < r < \infty$ and goes to zero quite rapidly as $r \to \infty$ or $r \to 2m$, i.e. as $r^* \to \pm\infty$,

$$\begin{aligned} r^* \to +\infty \Leftrightarrow r \to \infty : \quad & V_\ell(r) \sim (r^*)^{-2} \\ r^* \to -\infty \Leftrightarrow r \to 2m : \quad & V_\ell(r) \sim \mathrm{e}^{r^*/2m} \quad . \end{aligned} \tag{13.62}$$

This means that at infinity (in r) and near the horizon, the solutions of this equation can be chosen to have the standard right-moving (outgoing) / left-moving (ingoing) form

$$\psi(t,r^*) \sim \mathrm{e}^{\pm i\omega(t-r^*)} = \mathrm{e}^{\pm i\omega u} \quad \text{or} \quad \psi(t,r^*) \sim \mathrm{e}^{\pm i\omega(t+r^*)} = \mathrm{e}^{\pm i\omega v} \quad . \tag{13.63}$$

By separating out the time-dependence,

$$\psi(t,r^*) = \mathrm{e}^{-i\omega t}\psi(r^*) \ , \tag{13.64}$$

the exact equation to be solved takes the form of a standard time-independent Schrödinger equation,

$$-\partial_{r^*}^2\psi + V_\ell(r^*)\psi = \omega^2\psi \ . \tag{13.65}$$

It plays an important role in numerous aspects of Black Hole physics, e.g. in the analysys of the stability of the Schwarzschild solution. In this context, the above equation and its counterparts for vectors and symmetric tensors are known as the Regge-Wheeler(-Zerilli) equations.

13.9 The Kruskal-Szekeres Metric

Now let us to return to the exploration of the Schwarzschild geometry. Using Eddington-Finkelstein coordinates, we had discovered two "new" regions of the spacetime. Are there still other regions of space-time to be discovered? The answer is yes (as suggested by the analogy with Rindler and Minkowski space) and one way to find them would be to study spacelike rather than null geodesics. Alternatively, let us try to guess how one might be able to describe the maximal extension of space-time. The first guess might be to use the coordinates u and v simultaneously, instead of r and t. In these coordinates, the metric takes the form

$$ds^2 = -(1-2m/r)du\,dv + r^2 d\Omega^2 \ , \tag{13.66}$$

with $r = r(u,v)$. But while this is a good idea, the problem is that in these coordinates the horizon is once again infinitely far away, at $u = +\infty$ or $v = -\infty$ (i.e. at $2r^* = v - u = -\infty$). We can rectify this by introducing coordinates U and V with

$$U = -\mathrm{e}^{-u/4m} \quad , \quad V = \mathrm{e}^{v/4m} \tag{13.67}$$

say, so that the horizon is now at either $U = 0$ or $V = 0$. To better understand why we choose these coordinates and why exactly this factor in the exponent (and not any other positive number, which so far would have had the same effect of moving the horizon to $U = 0$ or $V = 0$), note that

$$\frac{v-u}{4m} = \frac{r}{2m} + \log\left(\frac{r}{2m} - 1\right) \ , \tag{13.68}$$

so that the prefactor $f(r) = 1 - 2m/r$ can be written as

$$1 - \frac{2m}{r} = \frac{2m}{r}\left(\frac{r}{2m} - 1\right) = \frac{2m}{r}\mathrm{e}^{-r/2m}\mathrm{e}^{(v-u)/4m} \quad . \tag{13.69}$$

Thus the metric can then be written as

$$ds^2 = \frac{2m}{r}\mathrm{e}^{-r/2m}\left(\mathrm{e}^{v/4m}dv\right)\left(-\mathrm{e}^{-u/4m}du\right) + r(u,v)^2 d\Omega^2 \quad . \tag{13.70}$$

It is now clear why (13.67) is a good choice: the metric now takes the simple form

$$ds^2 = -\frac{32m^3}{r}\mathrm{e}^{-r/2m}dUdV + r(U,V)^2 d\Omega^2 \quad , \tag{13.71}$$

with $r = r(U,V)$ is given implicitly by

$$UV = -\mathrm{e}^{(v-u)/4m} = -\mathrm{e}^{r^*/2m} = -(r/2m - 1)\mathrm{e}^{r/2m} \quad . \tag{13.72}$$

In these coordinates the metric is now manifestly completely non-singular and regular everywhere except at $r = 0$.

The horizon $r = 2m$ is mapped to $UV = 0$ which is the union of the two lines $U = 0$ and $V = 0$. Moreover, $t = t(U,V)$ is given by

$$U/V = -\mathrm{e}^{-(u+v)/4m} = -\mathrm{e}^{-t/2m} \quad , \tag{13.73}$$

The only remaining singularity is at $r = 0$ (this turns out to be not a mere coordinate singularity but a real singularity of the geometry - we will come back to this below).

Minor cosmetic improvements can be obtained by introducing, instead of U and V (13.67) the rescaled coordinates (u_K, v_K) through

$$\begin{aligned} du_K = \mathrm{e}^{-u/4m}du \quad &\Rightarrow \quad u_K = -4m\mathrm{e}^{-u/4m} = 4m\,U \\ dv_K = \mathrm{e}^{v/4m}dv \quad &\Rightarrow \quad v_K = +4m\mathrm{e}^{+v/4m} = 4m\,V \quad , \end{aligned} \tag{13.74}$$

in terms of which the metric takes the form

$$ds^2 = -\frac{2m}{r}\mathrm{e}^{-r/2m}du_K dv_K + r^2 d\Omega^2 \quad , \tag{13.75}$$

but then factors of $(16m^2)$ will reappear in other places, so we will forego this here. It is also possible to further scale u_K and v_K so that the exponential prefactor has the form $\exp(-(r-2m)/2m)$ (e.g. by defining r^* with a different integration constant, $r^* \to r^* - 2m$), so that for $r \to 2m$ the metric tends to $ds^2 \to -du_k dv_k$ without additional numerical factors like e^{-1} but, as I said, this is pure cosmetics. In any case, as we will learn later on, U and V are coordinates that are naturally defined only up to affine transformations, so one choice is as good as any other.

Finally, we pass from the null coordinates (U,V) (meaning that ∂_U and ∂_V are null vectors) to more familiar timelike and spacelike coordinates (T,X) defined, in anaogy with $(u,v) = t \mp r^*$, by

$$U = T - X \quad , \quad V = T + X \quad , \tag{13.76}$$

in terms of which the metric is

$$ds^2 = \frac{32m^3}{r}\mathrm{e}^{-r/2m}(-dT^2 + dX^2) + r^2 d\Omega^2 \quad . \tag{13.77}$$

Here $r = r(T, X)$ is now implicitly given by

$$X^2 - T^2 = (r/2m - 1)\mathrm{e}^{r/2m} \quad . \tag{13.78}$$

Chasing through the above sequence of coordinate transformations

$$(t, r) \to (t, r^*) \to (u, v) \to (U, V) \to (T, X) \quad , \tag{13.79}$$

one finds that the coordinate transformation $(t, r) \to (T, X)$ is explicitly, and in its full glory, given by

$$\begin{aligned} X(t, r) &= \tfrac{1}{2}(V - U) = (r/2m - 1)^{1/2}\mathrm{e}^{r/4m}\cosh t/4m \\ T(t, r) &= \tfrac{1}{2}(U + V) = (r/2m - 1)^{1/2}\mathrm{e}^{r/4m}\sinh t/4m \quad , \end{aligned} \tag{13.80}$$

As in Minkowski space, null lines are given by $X = \pm T + const$. The horizon is now at the null surfaces

$$r = 2m \quad \Rightarrow \quad X = \pm T \quad , \tag{13.81}$$

and surfaces of constant r are given by $X^2 - T^2 = const$.

Thus the original "Schwarzschild patch" $r > 2m$, the region of validity of the Schwarzschild coordinates, corresponds to the region $X > 0$ and $X^2 - T^2 > 0$, or $|T| < X$. As Figure 17 shows, this 'Schwarzschild patch' is mapped to the first quadrant of the Kruskal-Szekeres metric, bounded by the lines $X = \pm T$ which correspond to $r = 2m$.

But now that we have the coordinates X and T, we can let these coordinates (X, T) range over all the values for which the metric is non-singular. The only remaining singularity is at $r = 0$, which corresponds to the two sheets of the hyperboloid

$$r = 0 \quad \Leftrightarrow \quad T^2 - X^2 = 1 \quad \Leftrightarrow \quad T = \pm\sqrt{1 + X^2} \quad . \tag{13.82}$$

That $r = 0$ is indeed a real singularity that cannot be removed by a coordinate transformation can be shown by calculating some invariant of the curvature tensor, like the Kretschmann scalar $K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. On purely dimensional grounds, K must be proportional to m^2/r^6, the crucial feature being that the constant of proportionality is not zero, explicit calculations (this is a doable but thoroughly unenlightning exercise) showing that

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 48\frac{m^2}{r^6} \quad . \tag{13.83}$$

Thus the geometry is genuinely singular at $r = 0$. Nevertheless, since the metric is non-singular for all values of (X, T) subject to the constraint $r > 0$ or $T^2 - X^2 < 1$, there is no physical reason to exclude the regions in the other quadrants also satisfying this condition.

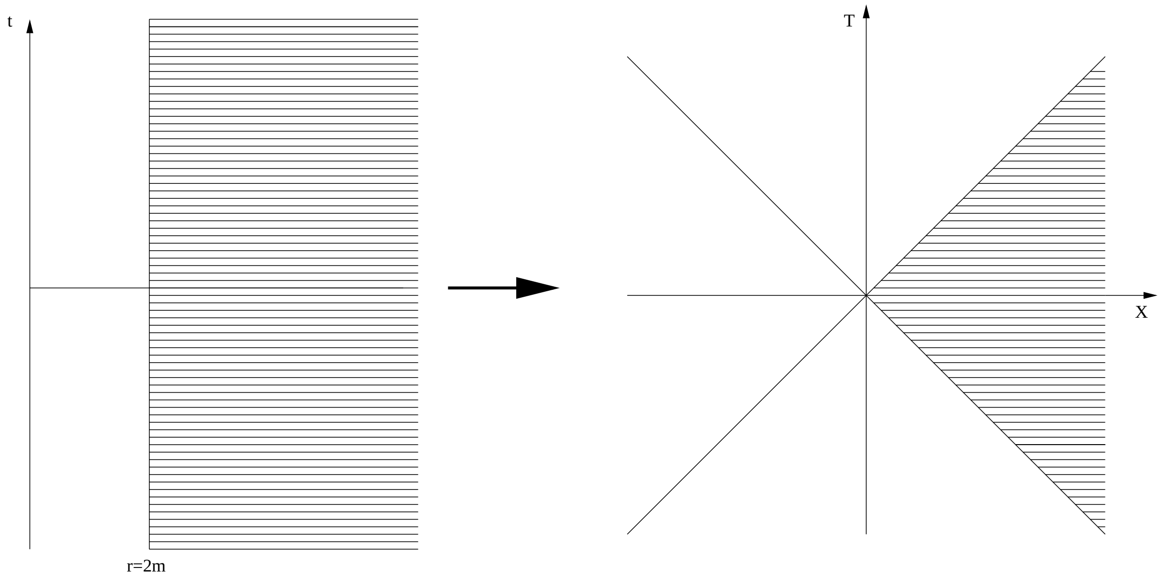

Figure 17: The Schwarzschild patch in the Kruskal-Szekeres metric: the half-plane $r > 2m$ is mapped to the quadrant between the lines $X = \pm T$ in the Kruskal-Szekeres metric.

By including them, we obtain the *Kruskal diagram* Figure 18. This extension of the Schwarzschild metric was discovered independently by Kruskal and Szekeres in 1960 and presents us with an amazingly rich and complex picture of what originally appeared to be a rather simple (and perhaps even dull) solution to the Einstein equations. It can be shown that this represents the maximal analytic extension of the Schwarzschild metric in the sense that every affinely parametrised geodesic can either be continued to infinite values of its parameter or runs into the singularity at $r = 0$ at some finite value of the affine parameter.[4]

In addition to the Schwarzschild patch, quadrant I, we have three other regions, living in the quadrants II, III, and IV, each of them having its own peculiarities. Note that obviously the conversion formulae from $(r, t) \to (X, T)$ in the quadrants II, III and IV differ from those given above for quadrant I. E.g. in region II one can use Schwarzschild(-like) coordinates in which the metric reads

$$ds^2 = \left(\frac{2m}{r} - 1\right) dt^2 - \left(\frac{2m}{r} - 1\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{13.84}$$

(these are not the same coordinates as those in patch I, as we have seen we cannot continue the Schwarzschild coordinates across the horizon), and in this quadrant (where r is a time coordinate etc.) the relation between Schwarzschild and Kruskal coordinates

[4]One can also show that any spherically symmetric solution of the Einstein equations is locally isometric to some domain of the Schwarzschild-Kruskal solution. For a proof of this generalised Birkhoff theorem see e.g. N. Straumann, *General Relativity*.

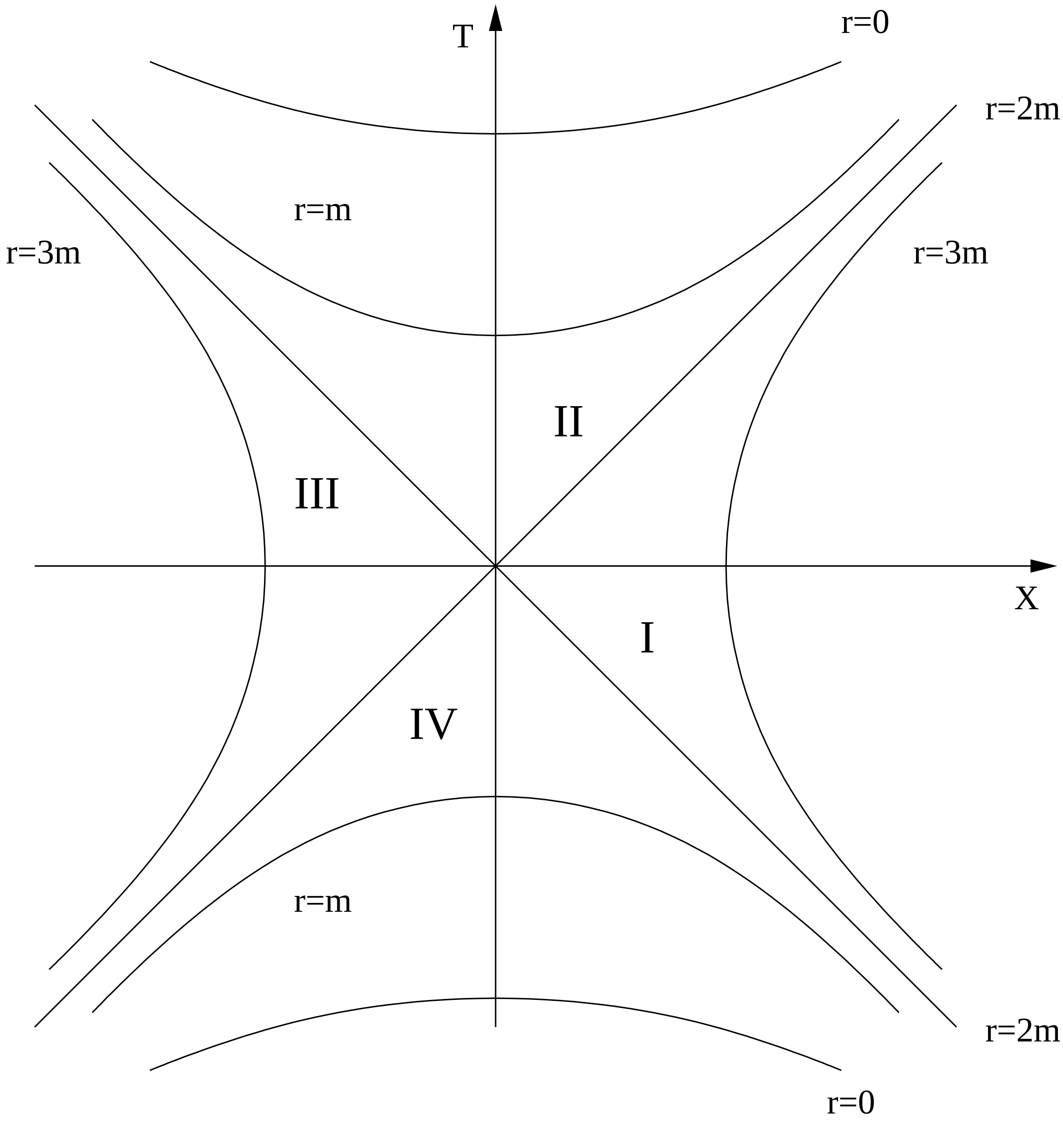

Figure 18: The complete Kruskal-Szekeres universe. Diagonal lines are null, lines of constant r are hyperbolas. Region I is the Schwarzschild patch, seperated by the horizon from regions II and IV. The Eddington-Finkelstein coordinates (u, r) cover regions I and II, (v, r) cover regions I and IV. Regions I and III are filled with lines of constant $r > 2m$. They are causally disconnected. Observers in regions I and III can receive signals from region IV and send signals to region II. An observer in region IV can send signals into both regions I and III (and therefore also to region II) and must have emerged from the singularity at $r = 0$ at a finite proper time in the past. Any observer entering region II will be able to receive signals from regions I and III (and therefore also from IV) and will reach the singularity at $r = 0$ in finite time. Events occuring in region II cannot be observed in any of the other regions.

is

$$\begin{aligned} X &= (1-r/2m)^{1/2}\mathrm{e}^{r/4m}\sinh t/4m \\ T &= (1-r/2m)^{1/2}\mathrm{e}^{r/4m}\cosh t/4m \ . \end{aligned} \tag{13.85}$$

To get acquainted with the Kruskal diagram, let us note the following basic facts:

1. Null lines are diagonals $X = \pm T + const.$, just as in Minkowski space. This greatly facilitates the exploration of the causal structure of the Kruskal-Szekeres metric.
2. In particular, the horizon corresponds to the two lines $X = \pm T$.
3. Lines of constant r are hyperbolas. For $r > 2m$ they fill the quadrants I and III, for $r < 2m$ the other regions II and IV.
4. In particular, the singularity at $r = 0$ is given by the two sheets of the hyperbola $T^2 - X^2 = 1$.
5. Notice in particular also that in regions II and IV worldlines with $r = const.$ are no longer timelike but spacelike.
6. Lines of constant Schwarzschild time t are straight lines through the origin. E.g. in region I one has $X = (\coth(t/4m))T$, with the future horizon $X = T$ corresponding, as expected, to $t \to \infty$.
7. The Eddington-Finkelstein coordinates (v, r) cover the regions I and II, the coordinates (u, r) the regions I and IV.
8. Quadrant III is completely new and is seperated from region I by a spacelike distance. That is, regions I and III are causally disconnected.

Now let us see what all this tells us about the physics of the Kruskal-Szekeres metric. An observer in region I (the familiar patch) can send signals into region II and receive signals from region IV. The same is true for an observer in the causally disconnected region III. Once an observer enters region II from, say, region I, he cannot escape from it anymore and he will run into the catastrophic region $r = 0$ in finite proper time. As a reward for his or her foolishness, between having crossed the horizon and being crushed to death, our observer will for the first time be able to receive signals and meet observers emerging from the mirror world in region III. Events occurring in region II cannot be observed anywhere outside that region (black hole). Finally, an observer in region IV must have emerged from the (past) singularity at $r = 0$ a finite proper time ago and can send signals and enter into either of the regions I or III.

Another interesting aspect of the Kruskal-Szekeres geometry is its dynamical character. This may appear to be a strange thing to say since we explicitly started off with a static metric. But this statement applies only to region I (and its mirror III). An investigation of the behaviour of spacelike slices analogous to that we performed at the end of section 11 for region I (see Figure 9) reveals a dynamical picture of continuing gravitational collapse in region II. In simple terms, the loss of staticity can be understood by noting that the timelike Killing vector field ∂_t of region I, when expressed in terms of Kruskal coordinates, becomes null on the horizon and spacelike in region II.

Indeed it is easy to check from (13.72) and (13.73) that the time-translation symnmetry $(t, r) \to (t + c, r)$ of the Schwarzschild patch corresponds to the transformation

$$U \to \mathrm{e}^{-c/4m} U \qquad V \to \mathrm{e}^{c/4m} V \ . \tag{13.86}$$

This is a boost in the (U, V) or (T, X)-plane which leaves the entire Kruskal metric invariant since $dUdV$ is invariant and $r = r(U, V)$ depends on U and V only via the boost-invariant quantity UV (13.72). Thus this symmetry is generated by the Killing vector

$$K = (V\partial_V - U\partial_U)/4m = (X\partial_T + T\partial_X)/4m \ , \tag{13.87}$$

It follows that K has norm proportional to $T^2 - X^2$,

$$||K||^2 = \frac{2m}{r}\mathrm{e}^{-r/2m}(T^2 - X^2) \ . \tag{13.88}$$

There are thus three different cases to consider, each of them interesting in its own right:

- K timelike

  K is timelike in the original region I (and in the mirror region III), with (13.78)

  $$I: \quad ||K||^2 = -\frac{2m}{r}\mathrm{e}^{-r/2m}(\frac{r}{2m} - 1)\mathrm{e}^{r/2m} = -(1 - \frac{2m}{r}) \ , \tag{13.89}$$

  confirming that $K = \partial_t$ in region I. We thus recover the statement that the Schwarzschild metric in the Schwarzschild patch is static.

- K spacelike

  K is spacelike in region II. Thus region II has no timelike Killing vector field, therefore cannot possibly be static, but has instead an additional spacelike Killing vector field.

  Related to this is the fact, already mentioned above, that in regions II and IV the slices of constant r are no longer timelike but spacelike surfaces. Thus they are analogous to, say, constant t or T slices for $r > 2m$. Just as it does not make sense to ask "where is the slice $t = 1$?" (say), only "when is $t = 1$?", or "where is $r = 3m$?", in these regions it makes no sense to ask "where is $r = m$?", only "when is $r = m$?".

- K null

  K is null on the horizon. This turns out to be the most interesting case, and therefore I will elaborate on this a bit in the following.

K is null on (and tangent to) the horizon $T = \pm X$ or $UV = 0$. In fact it reduces to $K = (V/4m)\partial_V = \partial_v$ on the horizon $U = 0$ (and to $K = \partial_u$ on $V = 0$). In this context (or from this perspective) the horizon is known as a *Killing horizon* (the locus where an asymptotically time-like Killing vector becomes null).

Therefore K generates translations along the horizon, $v \to v + c$, and is called the *null generator* of (this branch of) the horizon. Thus v can naturally be used as a coordinate there. Evidently, K vanishes at the "point" $U = V = 0$ where the other horizon $V = 0$ branches off. Actually this is of course not a point but a perfectly respectable 2-sphere of radius $r = 2m$ known as the *bifurcation surface* of the Killing horizon of the Schwarzschild geometry, and K vanishing means that this 2-sphere it is invariant under the time-translation generated by K, something that is also evident from (13.86). On $U = 0$ it lies at $V \to 0 \Rightarrow v \to -\infty$, so the Eddington-Finkelstein coordinate $v \in (-\infty, +\infty)$ only covers the half-line $U = 0, V > 0$ of the horizon.

On the other hand the line $U = 0$ is itself an "outgoing" (radial) null geodesic, but in light of the above v cannot possibly be an affine parameter along that geodesic (the affine parameter should not reach infinite values half-way along the geodesic). The failure of v to be an affine parameter on the horizon can be quantified by calculating the acceleration $\nabla_K K = \nabla_{\partial t}\partial_t$ (4.61) of the (integral curves of the) Killing vector and then taking the limit $r \to 2m$. This calculation is quite painless in Eddington-Finkelstein coordinates (v, r) in which $K = \partial_v$ everywhere,

$$K = \partial_t = (\partial_t v)\partial_v + (\partial_t r)\partial_r = \partial_v \tag{13.90}$$

(K is evidently a Killing vector because the components of the metric do not depend on v). Then the acceleration is

$$\nabla_K K \equiv (\nabla_K K)^\alpha \partial_\alpha = \Gamma^\alpha_{vv}\partial_\alpha = (f'/2)(f\partial_r + \partial_v) \tag{13.91}$$

where $f(r) = 1 - 2m/r$. Thus for $r \to 2m$ one finds

$$\lim_{r\to 2m} \nabla_K K = \left(\tfrac{1}{2}f'(r)|_{r=2m}\right) K = \left(\frac{m}{r^2}|_{r=2m}\right) K = \tfrac{1}{4m}K \ . \tag{13.92}$$

Since we have $\nabla_K K \sim K$ on the horizon, this shows first of all that there it generates a non-affinely parametrised geodesic (see (2.25) and (4.62)). Moreover, we see that one interpretation of the ubiquitous factor $1/4m$ is that it measures the failure of v to be an affine parameter on the horizon. Another interpretation, brought out by the same calculation, is that $1/4m$ is the *surface gravity* κ which measures the strength of the gravitational force (acceleration) $\mathsf{a}(r)$ (13.5) acting on a stationary observer at the horizon, but as measured at infinity (by taking into account the red-shift factor $f(r)^{1/2}$),

$$\begin{aligned} &\lim_{r\to 2m} \nabla_K K = \kappa K \\ &\kappa := \lim_{r\to 2m} f(r)^{1/2}\mathsf{a}(r) = \tfrac{1}{2}f'(r)|_{r=2m} = 1/4m \ . \end{aligned} \tag{13.93}$$

Note, incidentally, that the above calculation also shows that v is an affine parameter on a null surface at $r \to \infty$. Noting that v parametrises in-going light rays, this null surface is the surface of *past null infinity*, affectionately known as scri-minus (short for "script-I minus") $\mathcal{I}^-$. Likewise, u is an affine parameter on *future null infinity* $\mathcal{I}^+$.[5]

Anyway, to return to the beginning of this story, v is not an affine parameter along the horizon. It turns out, however, that the Kruskal coordinate V is an affine parameter there (and this is one way of understanding why Kruskal coordinates are so natural for exploring the causal structure of the metric), meaning that the null curve

$$x^\alpha(\lambda) = (U(\lambda), V(\lambda), \theta(\lambda), \phi(\lambda)) = (0, \lambda, \theta_0, \phi_0) \tag{13.94}$$

is an affinely parametrised null geodesic. This follows on the nose from (13.92) and the result (2.27) of section 2.2. Noting that κ is constant (actually not just along the geodesic but on the entire horizon, but the former is all we need), (2.27) with $\tau \to \lambda$ and $\sigma \to v$ becomes (dropping integration constants)

$$\frac{d\lambda}{dv} \sim \mathrm{e}^{\kappa v} \quad \rightarrow \quad \lambda(v) \sim \kappa^{-1}\mathrm{e}^{\kappa v} \sim V \ . \tag{13.95}$$

Another way to see this, which provides some more insight, is to analyse the geodesic Lagrangian and the conserved quantity associated to K in ingoing Eddington-Finkelstein coordinates (one could also work in Kruskal coordinates, but nothing is gained by that).

From the metric (13.48) we deduce that for a radial null-geodesics one has

$$-f(r)\dot{v}^2 + 2\dot{v}\dot{r} = 0 \tag{13.96}$$

where a dot denotes a derivative with respect to the affine parameter λ. The geodesics with $v = $ const describe ingoing null-geodesics and we are not interested in these, so we have

$$-f(r)\dot{v} + 2\dot{r} = 0 \ , \tag{13.97}$$

and since $u = t - r^* = v - 2r^*$ and $dr^*/dr = f(r)^{-1}$, this is equivalent to $\dot{u} = 0$ and, as anticipated, describes outgoing geodesics. Moreover, we have the conserved quantity E associated to the Killing vector $K = \partial_v$,

$$f\dot{v} - \dot{r} = E \ . \tag{13.98}$$

[5]Defining these objects properly requires singnifcantly more care. For details, and much more, see e.g. the classic text-book *The large scale structure of space-time* by S. Hawking and G. Ellis, or *General Relativity* by R. Wald.

From these two equations we immediately deduce (with a convenient parametrisation of the integration constant)

$$\dot{r} = E \quad \Rightarrow \quad r(\lambda) = E(\lambda - \lambda_0) + r_S \ . \tag{13.99}$$

For the null geodesic along the horizon we are ultimately interested in, we have $r(\lambda) = r_S$, i.e. $E = 0$, but we need to approach this with some care, so we keep the general solution for now. Analogously, for v we find the equation

$$f(r)\dot{v} = 2E \quad \Rightarrow \quad \dot{v} = \frac{2Er(\lambda)}{r(\lambda) - r_S} = 2E + \frac{2r_S}{\lambda - \lambda_0} \ . \tag{13.100}$$

We can now take the limit $E \to 0$ with impunity, and are left with

$$\dot{v} = \frac{2r_S}{\lambda - \lambda_0} \quad \Rightarrow \quad v(\lambda) = 2r_S \ln(\lambda - \lambda_0) + \text{const.} \ . \tag{13.101}$$

The prefactor $2r_S = 4m$ is now precisely such that it cancels the factor $1/4m$ in the definition of the Kruskal coordinate V (13.67), so that

$$V(\lambda) = \mathrm{e}^{v(\lambda)/4m} = a(\lambda - \lambda_0) \ , \tag{13.102}$$

which is precisely the statement that V is related to λ by an affine transformation. Thus V is an affine parameter along the horizon $U = 0$, as claimed.

We now have two natural coordinates on the future horizon $U = 0, V > 0$ which we can for instance use to measure the frequency of incoming waves. $\partial_t \to \partial_v$ measures what is commonly called Killing frequency (this requires no further explanation since it is associated to the Killing vector which generates time-translations in the Schwarzschild wedge), and ∂_V measures the so-called free fall frequency (since, more or less by construction, a freely falling observer in the Schwarzschild geometry near $r = 2m$ will see approximately the Minkowski space-time metric $ds^2 \sim -dUdV$). The exponential relation (13.67) between them reflects the exponential blue- or red-shift we first encountered in section 13.4.

In conclusion to this long excursion of this long section, I cannot resist mentioning that the surface gravity κ plays a crucial role in the analysis of the classical dynamics of black holes, and even more so in the semi-classical context, since it is directly proportional to the temperature of the famous Hawking radiation of an evaporating black hole,

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{8\pi GM} \ . \tag{13.103}$$

For more details on this and other related advanced topics I am not able to cover or do justice to here, I refer you to the superb Cambridge lecture notes on Black Holes.[6]

[6] P. Townsend, *Black Holes*, `arXiv:gr-qc/9707012v1`.

Now you may well wonder if all this talk about white holes and mirror regions is for real or just science fiction. Clearly, if an object with $r_0 < 2m$ exists and is described by the Schwarzschild solution, then we will have to accept the conclusions of the previous section. However, this requires the existence of an eternal black hole (in particular, eternal in the past) in an asymptotically flat space-time, and this is not very realistic. While black holes are believed to exist, they are believed to form as a consequence of the gravitational collapse of a star whose nuclear fuel has been exhausted (and which is so massive that it cannot settle into a less singular final state like a *White Dwarf* or *Neutron Star*).

To see how we could picture the situation of gravitational collapse (without trying to understand why this collapse occurs in the first place), let us estimate the average density ρ of a star whose radius r_0 is equal to its Schwarzschild radius. For a star with mass M we have

$$r_S = \frac{2MG}{c^2} \tag{13.104}$$

and approximately

$$M = \frac{4\pi r_0^3}{3}\rho \ . \tag{13.105}$$

Therefore, setting $r_0 = r_S$, we find that

$$\rho = \frac{3c^6}{32\pi G^3 M^2} \approx 2\times 10^{16}\ \mathrm{g/cm^3}\ \left(\frac{M_{\rm sun}}{M}\right)^2 \ . \tag{13.106}$$

For stars of a few solar masses, this density is huge, roughly that of nuclear matter. In that case, there will be strong non-gravitational forces and hydrodynamic processes, singificantly complicating the description of the situation. The situation is quite simple, however, when an object of the mass and size of a galaxy ($M \sim 10^{10} M_{\rm sun}$) collapses. Then the critical density (13.106) is approximately that of air, $\rho \sim 10^{-3}$ g/cm^3, non-gravitational forces can be neglected completely, and the collapse of the object can be approximated by a free fall. The Schwarzschild radius of such an object is of the order of light-days ($\sim 10^5$s).

Under these circumstances, a more realistic Kruskal-like space-time diagram of a black hole would be the one depicted in Figure 19. We assume that at time $t = 0$ ($T = 0$) we have a momentarily static mass configuration with radius $R \gg 2m$ and mass M which then starts to collapse in free fall. Neglecting radiation-effects, the mass M of the star (galaxy) remains constant so that the exterior of the star $r > R$ is described by the corresponding subset of region I of the Kruskal-Szekeres metric. Note that regions III and IV no longer exist because the region $r < R$ is simply not at all described by the Schwarzschild solution, but should be described by a solution of the Einstein equations appropriate for the interior of the star (in particular, this better be a solution of the non-vacuum Einstein equations).

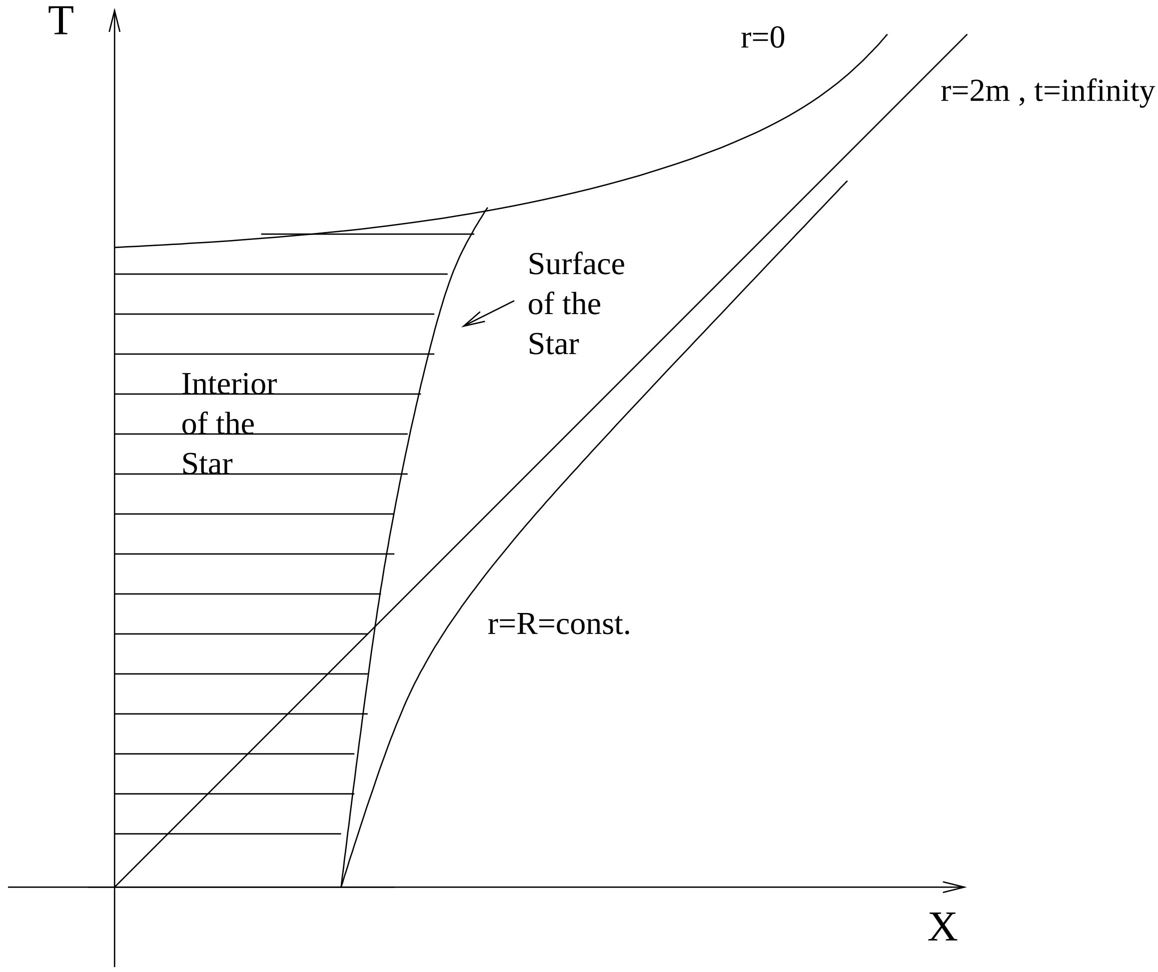

Figure 19: The Kruskal diagram of a gravitational collapse. The surface of the star is represented by a timelike geodesic, modelling a star (or galaxy) in free fall under its own gravitational force. The surface will reach the singularity at $r = 0$ in finite proper time whereas an outside observer will never even see the star collapse beyond its Schwarzschild radius. However, as discussed in the text, even for an outside observer the resulting object is practically 'black'.

The surface of the star can be represented by a timelike geodesic going from $r = R$ at $t = 0$ to $r = 0$. According to (13.13), it will reach $r = 0$ after the finite proper time

$$\tau_{R\to 0} = 2\left(\frac{R^3}{2m}\right)^{1/2} \int_0^{\pi/2} d\alpha\, \sin^2\alpha = \pi \left(\frac{R^3}{8m}\right)^{1/2} \quad . \tag{13.107}$$

For an object the size of the sun this is of the order of one hour!

Note that, even if the free fall (geodesic) approximation is no longer justified at some point, once the surface of the star has crossed the Schwarzschild horizon, nothing, no amount of pressure, can stop the catastrophic collapse to $r = 0$ because, whatever happens, points on the surface of the star will have to move within their forward lightcone and will therefore inevitably end up at $r = 0$ (and since timelike geodesics *maximise* proper time, any non-geodesic attempt to avoid hitting $r = 0$ will only get you there even quicker ...).

In interepreting this it should be kept in mind that the Schwarzschild metric was never meant to be valid at $r = 0$ anyway (as it is supposed to describe the exterior of a gravitating body). Nevertheless, just being close enough to $r = 0$, without actually reaching that point is more than sufficient to crush any kind of matter. Indeed, (13.83) and the geodesic deviation equation (section 8.3) show that the force needed to keep neighbouring particles apart is proportional to r^{-3}. Thus the tidal forces within arbitrary objects (be they solids or elementary particles) eventually become infinitely big so that these objects will be crushed completely. In that sense, the physics becomes hopelessly singular even before one reaches $r = 0$ and there seems to be nothing to prevent a collapse of such an object to $r = 0$ and infinite density. Certainly classical general relativity (and even current-day quantum field theory) are inadequate to describe this situation (and if or how a theory of quantum gravity can deal with these matters remains to be seen).

For an observer remaining outside the collpasing star, say at the constant value $r = r_\infty$, the situation (not unexpectedly by now) presents itself in a rather different way. Up to a constant factor $(1 - 2m/r_\infty)^{1/2}$, his proper time equals the coordinate time t. As the surface of the collapsing galaxy crosses the horizon at $t = \infty$, strictly speaking the outside observer will never see the black hole form.

We had already encountered a similar phenomenon in our discussion of the infinite gravitational red-shift. As we have seen, the gravitational red-shift grows exponentially with time (13.28), $z \sim \exp t/4m$, for radially emitted photons. The luminosity L of the star decreases exponentially, as a consequence of the gravitational red-shift and the fact that photons emitted at equal time intervals from the surface of the star reach the observer at greater and greater time intervals. It can be shown that

$$L \sim \mathrm{e}^{-t/3\sqrt{3}m} \quad , \tag{13.108}$$

so that the star becomes very dark very quickly, the characteristic time being of the order of

$$3\sqrt{3}m \approx 2,5 \times 10^{-5}\text{s} \ \left(\frac{M}{M_{\text{sun}}}\right) \ . \qquad (13.109)$$

Thus, even though for an outside observer the collapsing star never disappears completely, for all practical intents and purposes the star is black and the name 'black hole' is justified.

It is fair to wonder at this point if the above conclusions regarding the collapse to $r = 0$ are only a consequence of the fact that we assumed exact spherical symmetry. Would the singularity be avoided under more general conditions? The answer to this is, somewhat surprisingly and shockingly, a clear 'no'.

There are very general *singularity theorems*, due to Penrose, Hawking and others, which all state in one way or another that if Einstein's equations hold, the energy-momentum tensor satisfies some kind of positivity condition, and there is a regular event horizon, then some kind of singularity will appear. These theorems do not rely on any symmetry assumptions.

It has also been shown that the gravitational field of a static black hole, even without further symmetry assumptions, is necessarily given by the spherically symmetric Schwarzschild metric and is thus characterised by the single parameter M.

Of course, other exact solutions describing isolated systems like a star, meaning that the solution is asymptotically flat, are known. Two important examples are the following:

1. The Reissner-Nordstrøm Metric

   The Reissner-Nordstrøm metric is a solution to the coupled Einstein-Maxwell equations describing the gravitational field of a spherically symmetric electrically charged star. It is characterised by two parameters, its mass M and its charge Q, with $F_{tr} = -Q/r^2$, and the metric is

   $$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + (1 - \frac{2M}{r} + \frac{Q^2}{r^2})^{-1}dr^2 + r^2 d\Omega^2 \ . \qquad (13.110)$$

   Note that this can be obtained from the Schwarzschild metric by substituting

   $$M \to M - \frac{Q^2}{2r} \ . \qquad (13.111)$$

   The structure of the singularities and event horizons is more complicated now than in the case of the Schwarzschild metric and also depends on the relative size of Q and M.

   If $Q^2 > M^2$ (this is not a very realistic situation), then the metric is non-singular everywhere except, of course, at $r = 0$. In particular, the coordinate t is always timelike and the coordinate r is always spacelike. While this may sound quite pleasing, much less insane than what happens for the Schwarzschild metric, this is actually a disaster. The singularity at $r = 0$ is now timelike, and it is not protected by an event-horizon. Such a singularity is known as a *naked singularity.* An observer could travel to the singularity and come back again. Worse, whatever happens at the singularity can influence the future physics away from the singularity, but as there is a singularity this means that the future cannot be predicted/calculated in such a space-time because the laws of physics break down at $r = 0$. There is a famous conjecture, known as the *Cosmic Censorship Conjecture*, which roughly speaking states that the collapse of physically realistic matter configurations will generically not lead to a naked singularity. In spite of a lot of partial results and circumstantial evidence in favour of this conjecture, it is not known if (or in which precise form) it holds in General Relativity.

The situation is even more interesting in the somewhat more realistic case $M^2 > Q^2$. In that case, there are two radii

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2} \qquad (13.112)$$

at which the metric becomes singular. The outer one is just like the event horizon of the Schwarzschild metric, the inner one reverses the role of radius and time once more so that the singularity is timelike and can be avoided by returning to larger values of r. There is much more that can and should be said about this solution but I will not do this here.

2. The Kerr Metric

   The Kerr metric describes a rotating black hole and is characterised by its mass M and its angular momentum J. Now one no longer has spherical symmetry (because the axis of rotation picks out a particular direction) but only axial symmetry. The situation is thus a priori much more complicated. A stationary solution (i.e. one with a timelike Killing vector, 'static' is a slightly stronger condition) was found by Kerr only in 1963, almost fifty years after the Schwarzschild and Reissner-Nordstrøm solutions. Its singularity and horizon structure is much more intricate and intriguing than that of the solutions discussed before. One can pass from one universe into a different asymptotically flat universe. The singularity at $r = 0$ has been spread out into a ring; if one enters into the ring, one can not only emerge into a different asymptotically flat space-time but one can also turn back in time (there are closed timelike curves), one can dip into the black hole and emerge with more energy than one had before (at the expense of the angular momentum of the black hole), etc. etc. All this is fun but also rather technical and I will not go into any of this here.

   However, it is worth keeping in mind that the Kerr metric is definitely of astrophysical importance. Astrophysical black holes, while they may carry negligible charge Q, are expected to typically have a non-zero angular momentum J.

Of course there are also solutions describing a combination of the two above solutions, namely charged rotating black holes (the Kerr-Newman metric). One of the reasons why I mention these solutions is that it can be shown that the most general stationary electrically charged black hole is characterised by just three parameters, namely M, Q and J. This is generally referred to as the fact that *black holes have no hair*, or as the *no-hair theorem*. It roughly states that the only characteristics of a black hole which are not somehow radiated away during the phase of collapse via multipole moments of the gravitational, electro-magnetic, ... fields are those which are protected by some conservation laws, something that in simple cases can be confirmed by an explicit calculation.

14 Interlude: Maximally Symmetric Spaces

As a preparation for our discussion of cosmology in subsequent sections, in this section we will discuss some aspects of what are known as *maximally symmetric spaces.* These are spaces that admit the maximal number of Killing vectors (which turns out to be $n(n+1)/2$ for an n-dimensional space). As we will discuss later on, in the context of the *Cosmological Principle*, such spaces, which are simultaneously homogeneous ("the same at every point") and isotropic ("the same in every direction") provide an (admittedly highly idealised) description of space in a cosmological space-time,

14.1 Curvature and Killing Vectors

In order to understand how to define and characterise maximally symmetric spaces, we will need to obtain some more information about how Killing vectors can be classified.

For reasons that will become apparent below, we will first derive an identity involving Killing vectors and the curvature tensor. Using the defining relation of the Riemann curvature tensor,

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V_\rho = -R^\lambda_{\ \rho\mu\nu} V_\lambda \tag{14.1}$$

and its cyclic symmetry,

$$R^\lambda_{\ \rho\mu\nu} + R^\lambda_{\ \nu\rho\mu} + R^\lambda_{\ \mu\nu\rho} = 0 \ \ , \tag{14.2}$$

it is possible to deduce (exercise!) that for a Killing vector K^μ, $\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0$, one has

$$\nabla_\lambda \nabla_\mu K_\nu(x) = R^\rho_{\ \lambda\mu\nu}(x) K_\rho(x) \ \ . \tag{14.3}$$

This has a rather remarkable consequence: as you can see, the second derivatives of the Killing vector at a point x_0 are again expressed in terms of the value of the Killing vector itself at that point. But this means (think of Taylor expansions), that a Killing vector field $K^\mu(x)$ is completely determined everywhere by the values of $K_\mu(x_0)$ and $\nabla_\mu K_\nu(x_0)$ at a single point x_0.

A set of Killing vectors $\{K^{(i)}_\mu(x)\}$ is said to be *independent* if any linear relation of the form

$$\sum_i c_i K^{(i)}_\mu(x) = 0 \ \ , \tag{14.4}$$

with *constant* coefficients c_i implies $c_i = 0$.

Since, in an n-dimensional space(-time) there can be at most n linearly independent vectors $(K_\mu(x_0))$ at a point, and at most $n(n-1)/2$ independent anti-symmetric matrices $(\nabla_\mu K_\nu(x_0))$, we reach the conclusion that an n-dimensional space(-time) can have at most

$$n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2} \tag{14.5}$$

independent Killing vectors. An example of a metric with the maximal number of Killing vectors is, none too surprisingly, n-dimensional Minkowski space, where $n(n+1)/2$ agrees with the dimension of the Poincaré group, the group of transformations that leave the Minkowski metric invariant.

As an aside, note that contracting (14.3) over λ and μ, one learns that

$$\nabla^\mu \nabla_\mu K_\nu = -R_{\nu\mu} K^\mu \quad . \tag{14.6}$$

In particular, if (or wherever) $R_{\mu\nu} = 0$, the antisymmetric tensor

$$A_{\mu\nu} = \nabla_\mu K_\nu = -A_{\nu\mu} \tag{14.7}$$

is covariantly conserved,

$$R_{\mu\nu} = 0 \quad \Rightarrow \quad \nabla^\mu A_{\mu\nu} = 0 \Leftrightarrow \partial_\mu(\sqrt{g} A^\mu{}_\nu) = 0 \quad . \tag{14.8}$$

As this is now, unlike an energy-momentum tensor (cf. the discussion in section 5.7), an antisymmetric tensor, this allows one e.g. to construct corresponding conserved charges, something along the lines of

$$Q_K = \int A_{\mu\nu} d\Sigma^{\mu\nu} \quad , \tag{14.9}$$

associated to isometries of solutions of the vacuum Einstein equations. These are known as *Komar charges.*

14.2 Homogeneous, Isotropic and Maximally Symmetric Spaces

We have seen above that Killing vectors $K^\mu(x)$ are determined by the values $K^\mu(x_0)$ and $\nabla_\mu K_\nu(x_0)$ at a single point x_0. We will now see how these data are related to translations and rotations.

We define a *homogeneous space* to be such that it has infinitesimal isometries that carry any given point x_0 into any other point in its immediate neighbourhood (this could be stated in more fancy terms!). Thus the metric must admit Killing vectors that, at any given point, can take all possible values. Thus we require the existence of Killing vectors for arbitrary $K_\mu(x_0)$. This means that the n-dimensional space admits n *translational Killing vectors.*

We define a space to be *isotropic at a point* x_0 if it has isometries that leave the given point x_0 fixed and such that they can rotate any vector at x_0 into any other vector at x_0. Therefore the metric must admit Killing vectors such that $K_\mu(x_0) = 0$ but such that $\nabla_\mu K_\nu(x_0)$ is an arbitrary antisymmetric matrix (for instance to be thought of as an element of the Lie algebra of $SO(n)$). This means that the n-dimensional space admits $n(n-1)/2$ *rotational Killing vectors.*

Finally, we define a *maximally symmetric space* to be a space with a metric with the maximal number $n(n+1)/2$ of Killing vectors.

Some simple and fairly obvious consequences of these definitions are the following:

1. A homogeneous and isotropic space is maximally symmetric.
2. A space that is isotropic for all x is also homogeneous. (This follows because linear combinations of Killing vectors are again Killing vectors and the difference between two rotational Killing vectors at x and $x + dx$ can be shown to be a translational Killing vector.)
3. (1) and (2) now imply that a space which is isotropic around every point is maximally symmetric.
4. Finally one also has the converse, namely that a maximally symmetric space is homogeneous and isotropic.

In practice the characterisation of a maximally symmetric space which is easiest to use is (3) because it requires consideration of only one type of symmetries, namely rotational symmetries.

14.3 The Curvature Tensor of a Maximally Symmetric Space

On the basis of these simple considerations we can already determine the form of the Riemann curvature tensor of a maximally symmetric space. We will see that maximally symmetric spaces are spaces of constant curvature in the sense that

$$R_{ijkl} = k(g_{ik}g_{jl} - g_{il}g_{jk}) \tag{14.10}$$

for some constant k.

This result could be obtained by making systematic use of the higher order integrability conditions for the existence of a maximal number of Killing vectors. The argument given below is less covariant but more elementary.

Assume for starters that the space is isotropic at x_0 and choose a Riemann normal coordinate system centered at x_0. Thus the metric at x_0 is $g_{ij}(x_0) = \eta_{ij}$ where we may just as well be completely general and assume that

$$\eta_{ij} = \text{diag}(\underbrace{-1,\ldots,-1}_{p\text{ times}}\underbrace{+1,\ldots,+1}_{q\text{ times}}) \ , \tag{14.11}$$

where $p + q = n$ and we only assume $n > 2$.

If the metric is supposed to be isotropic at x_0 then, in particular, the curvature tensor at the origin must be invariant under Lorentz rotations. Now we know (i.e. you should know from your Special Relativity course) that the only invariants of the Lorentz group are the Minkowski metric and products thereof, and the totally antisymmetric epsilon-symbol. Thus the Riemann curvature tensor has to be of the form

$$R_{ijkl}(x_0) = a\eta_{ij}\eta_{kl} + b\eta_{ik}\eta_{jl} + c\eta_{il}\eta_{jk} + d\epsilon_{ijkl} \quad , \tag{14.12}$$

where the last term is only possible for $n = 4$. The symmetries of the Riemann tensor imply that $a = d = b + c = 0$, and hence we are left with

$$R_{ijkl}(x_0) = b(\eta_{ik}\eta_{jl} - \eta_{il}\eta_{jk}) \quad , \tag{14.13}$$

Thus in an arbitrary coordinate system we will have

$$R_{ijkl}(x_0) = b(g_{ik}(x_0)g_{jl}(x_0) - g_{il}(x_0)g_{jk}(x_0)) \quad , \tag{14.14}$$

If we now assume that the space is isotropic around every point, then we can deduce that

$$R_{ijkl}(x) = b(x)(g_{ik}(x)g_{jl}(x) - g_{il}(x)g_{jk}(x)) \tag{14.15}$$

for some function $b(x)$. Therefore the Ricci tensor and the Ricci scalar are

$$\begin{aligned} R_{ij}(x) &= (n-1)b(x)g_{ij} \\ R(x) &= n(n-1)b(x) \quad . \end{aligned} \tag{14.16}$$

and the Riemann curvature tensor can also be written as

$$R_{ijkl} = \frac{R}{n(n-1)}(g_{ik}g_{jl} - g_{il}g_{jk}) \quad , \tag{14.17}$$

while the Einstein tensor is

$$G_{ij} = b[(n-1)(1-n/2)]g_{ij} \quad . \tag{14.18}$$

The contracted Bianchi identity $\nabla^i G_{ij} = 0$ now implies that $b(x)$ has to be a constant, and we have thus established (14.10). Note that we also have

$$R_{ij} = k(n-1)g_{ij} \quad , \tag{14.19}$$

so that a maximally symmetric space(-time) is automatically a solution to the vacuum Einstein equations with a cosmological constant. In the physically relevant case $p = 1$ these are known as de Sitter or anti de Sitter space-times. We will come back to them later on. In general, solutions to the equation $R_{ij} = cg_{ij}$ for some constant c are known as *Einstein manifolds* in the mathematics literature.

14.4 The Metric of a Maximally Symmetric Space I

We are interested not just in the curvature tensor of a maximally symmetric space but in the metric itself. I will give you two derivations of the metric of a maximally symmetric space, one by directly solving the differential equation

$$R_{ij} = k(n-1)g_{ij} \tag{14.20}$$

for the metric g_{ij}, the other by a direct geometrical construction of the metric which makes the isometries of the metric manifest.

As a maximally symmetric space is in particular spherically symmetric, we already know that we can write its metric in the form

$$ds^2 = B(r)dr^2 + r^2 d\Omega^2_{(n-1)} \quad , \tag{14.21}$$

where $d\Omega^2_{(n-1)} = d\theta^2 + \ldots$ is the volume-element for the $(n-1)$-dimensional sphere or its counterpart in other signatures. For concreteness, we now fix on $n = 3$, but the argument given below goes through in general.

We have already calculated all the Christoffel symbols for such a metric (set $A(r) = 0$ in the calculations leading to the Schwarzschild metric in section 11), and we also know that $R_{ij} = 0$ for $i \neq j$ and that all the diagonal angular components of the Ricci tensor are determined by $R_{\theta\theta}$ by spherical symmetry. Hence we only need R_{rr} and $R_{\theta\theta}$, which are

$$\begin{aligned} R_{rr} &= \frac{1}{r}\frac{B'}{B} \\ R_{\theta\theta} &= -\frac{1}{B} + 1 + \frac{rB'}{2B^2} \quad , \end{aligned} \tag{14.22}$$

and we want to solve the equations

$$\begin{aligned} R_{rr} &= 2kg_{rr} = 2kB(r) \\ R_{\theta\theta} &= 2kg_{\theta\theta} = 2kr^2 \quad . \end{aligned} \tag{14.23}$$

From the first equation we obtain

$$B' = 2krB^2 \quad , \tag{14.24}$$

and from the second equation we deduce

$$\begin{aligned} 2kr^2 &= -\frac{1}{B} + 1 + \frac{rB'}{2B^2} \\ &= -\frac{1}{B} + 1 + \frac{2kr^2B^2}{2B^2} \\ &= -\frac{1}{B} + 1 + kr^2 \quad . \end{aligned} \tag{14.25}$$

This is an algebraic equation for B solved by

$$B = \frac{1}{1-kr^2} \tag{14.26}$$

(and this also solves the first equation). Therefore we have determined the metric of a a maximally symmetric space to be

$$ds^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2_{(n-1)} \quad . \tag{14.27}$$

Let us pass back from polar coordinates to Cartesian coordinates, with $r^2 = \vec{x}^2 = \eta_{ij}x^i x^j$. Then we have $rdr = \vec{x}.d\vec{x}$ and $dr^2 = (\vec{x}.d\vec{x})^2/\vec{x}^2$. Hence this metric can also be written as

$$ds^2 = d\vec{x}^2 + \frac{k(\vec{x}.d\vec{x})^2}{1-k\vec{x}^2} \quad . \tag{14.28}$$

Clearly, for $k=0$ this is just the flat metric on $\mathbb{R}^{p,q}$. For $k=1$, this should also look familiar as the standard metric on the sphere. If not, don't worry. We will discuss the $k \neq 0$ metrics in more detail in the next section. This will make the isometries of the metric manifest and will also exclude the possibility, not logically ruled out by the arguments given so far, that the metrics we have found here for $k \neq 0$ are spherically symmetric and have constant Ricci curvature but are not actually maximally symmetric.

14.5 The Metric of a Maximally Symmetric Space II

Recall that the standard metric on the n-sphere can be obtained by restricting the flat metric on an ambient $\mathbb{R}^{n+1}$ to the sphere. We will generalise this construction a bit to allow for $k<0$ and other signatures as well.

Consider a flat auxiliary vector space V of dimension $(n+1)$ with metric

$$ds^2 = d\vec{x}^2 + \frac{1}{k}dz^2 \quad , \tag{14.29}$$

whre $\vec{x} = (x^1, \ldots, x^n)$ and $d\vec{x}^2 = \eta_{ij}dx^i dx^j$. Thus the metric on V has signature $(p, q+1)$ for k positive and $(p+1, q)$ for k negative. The group $G = SO(p, q+1)$ or $G = SO(p+1, q)$ has a natural action on V by isometries of the metric. The full isometry group of V is the semi-direct product of this group with the Abelian group of translations (just as in the case of the Euclidean or Poincaré group).

Now consider in V the hypersurface Σ defined by

$$k\vec{x}^2 + z^2 = 1 \quad . \tag{14.30}$$

This equation breaks all the translational isometries, but by the very definition of the group G it leaves this equation, and therefore the hypersurface Σ. It follows that G will act by isometries on Σ with its induced metric. But $\dim G = n(n+1)/2$. Hence the n-dimensional space has $n(n+1)/2$ Killing vectors and is therefore maximally symmetric.

In fact, G acts transitively on Σ (thus Σ is homogeneous) and the stabilizer at a given point is isomorphic to $H = SO(p,q)$ (so Σ is isotropic), and therefore Σ can also be described as the homogeneous space

$$\begin{aligned} \Sigma_{k>0} &= SO(p,q+1)/SO(p,q) \\ \Sigma_{k<0} &= SO(p+1,q)/SO(p,q) \ . \end{aligned} \tag{14.31}$$

The Killing vectors of the induced metric are simply the restriction to Σ of the standard generators of G on the vector space V.

It just remains to determine explicitly this induced metric. For this we start with the defining relation of Σ and differentiate it to find that on Σ one has

$$dz = -\frac{k\vec{x}.d\vec{x}}{z} \ , \tag{14.32}$$

so that

$$dz^2 = \frac{k^2(\vec{x}.d\vec{x})^2}{1-k\vec{x}^2} \ . \tag{14.33}$$

Thus the metric (14.29) restricted to Σ is

$$\begin{aligned} ds^2|_\Sigma &= d\vec{x}^2 + \frac{1}{k}dz^2|_\Sigma \\ &= d\vec{x}^2 + \frac{k(\vec{x}.d\vec{x})^2}{1-k\vec{x}^2} \ . \end{aligned} \tag{14.34}$$

This is precisely the same metric as we obtained in the previous section.

For Euclidean signature, these spaces are spheres and hyperspheres (hyperboloids), and in other signatures they are the corresponding generalisations. In particular, for $(p,q) = (1, n-1)$ we obtain *de Sitter space-time* for $k = 1$ and *anti de Sitter space-time* for $k = -1$. They have the topology of $S^{n-1} \times \mathbb{R}$ and $\mathbb{R}^{n-1} \times S^1$ respectively and, as mentioned before, they solve the vacuum Einstein equations with a positive (negative) cosmological constant.

14.6 The Metric of a Maximally Symmetric Space III

Finally, it will be useful to see the maximally symmetric metrics in some other coordinate systems. For $k = 0$, there is nothing new to say since this is just the flat metric. Thus we focus on $k \neq 0$.

First of all, let us note that essentially only the sign of k matters as $|k|$ only effects the overall size of the space and nothing else (and can therefore be absorbed in the scale factor $a(t)$ of the metric (15.1)). To see this note that by rescaling of r, $r' = |k|^{1/2}r$, the metric (14.27) can be put into the form

$$ds^2 = \frac{1}{|k|}\left(\frac{dr'^2}{1 \pm r'^2} + r'^2 d\Omega^2_{(n-1)}\right) \ . \tag{14.35}$$

Thus we will just need to consider the cases $k = \pm 1$.

For $k = +1$, we have

$$ds^2 = \frac{dr^2}{1 - r^2} + r^2 d\Omega^2_{(n-1)} \quad . \tag{14.36}$$

Thus, obviously the range of r is restricted to $r \leq 1$ and by the change of variables $r = \sin\psi$, the metric can be put into the standard form of the metric on S^n in polar coordinates,

$$ds^2 = d\psi^2 + \sin^2\psi d\Omega^2_{(n-1)} \quad . \tag{14.37}$$

This makes it clear that the singularity at $r = 1$ is just a coordinate singularity. It would also appear if one wrote the metric on the two-sphere in terms of the radial coordinate $r = \sin\theta$,

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 = \frac{dr^2}{1 - r^2} + r^2 d\phi^2 \quad . \tag{14.38}$$

For $k = -1$, on the other hand, we have

$$ds^2 = \frac{dr^2}{1 + r^2} + r^2 d\Omega^2_{(n-1)} \quad . \tag{14.39}$$

Thus the range of r is $0 \leq r < \infty$, and we can use the change of variables $r = \sinh\psi$ to write the metric as

$$ds^2 = d\psi^2 + \sinh^2\psi d\Omega^2_{(n-1)} \quad . \tag{14.40}$$

This is the standard metric of a hyperboloid H^n in polar coordinates.

Finally, by making the change of variables

$$r = \bar{r}(1 + k\bar{r}^2/4)^{-1} \quad , \tag{14.41}$$

one can put the metric in the form

$$ds^2 = (1 + k\bar{r}^2/4)^{-2}(d\bar{r}^2 + \bar{r}^2 d\Omega^2_{(n-1)}) \quad . \tag{14.42}$$

Note that this differs by the *conformal factor* $(1 + k\bar{r}^2/4)^{-2} > 0$ from the flat metric. One says that such a metric is *conformally flat.* Thus what we have shown is that every maximally symmetric space is conformally flat. Note that conformally flat, on the other hand, does not imply maximally symmetric as the conformal factor could also be any function of the radial and angular variables.

15 Cosmology I: Basics

15.1 Preliminary Remarks

We now turn away from considering isolated systems (stars) to some (admittedly very idealised) description of the universe as a whole. This subject is known as Cosmology. It is certainly one of the most fascinating subjects of theoretical physics, dealing with such issues as the origin and ultimate fate and the large-scale structure of the universe.

Due to the difficulty of performing cosmological experiments and making precise measurements at large distances, many of the most basic questions about the universe are still unanswered today:

1. Is our universe open or closed?
2. Will it keep expanding forever or will it recollapse?
3. Why is the Cosmic Microwave Background radiation so isotropic?
4. What is the mechanism responsible for structure formation in the universe?
5. Where and what is the 'missing mass' (Dark Matter)?
6. Why is the cosmological constant so small and what is its value?
7. Is Dark Energy, responsible for what appears to be a current phase of acceleration of the universe, a cosmological constant?

While recent precision data, e.g. from supernovae surveys and detailed analysis of the cosmic microwave background radiation, suggest answers to at least some of these questions, these answers actually just make the universe more myseterious than ever.

Of course, we cannot study any of these questions in detail, in particular because an important role in studying these questions is played by the interaction of cosmology with astronomy, astrophysics and elementary particle physics, each of these subjects deserving at least a course of its own.

Fortunately, however, many of the important features any realistic cosmological model should display are already present in some very simple models, the so-called *Friedmann-Robertson-Walker Models* already studied in the 20's and 30's of the last century. They are based on the simplest possible ansatz for the metric compatible with the assumption that on large scales the universe is roughly homogeneous and isotropic (cf. the next section for a more detailed discussion of this *Cosmological Principle*) and have become the 'standard model' of cosmology.

We will see that they already display all the essential features such as

1. a Big Bang
2. expanding universes (Hubble expansion)
3. different long-term behaviour (eternal expansion versus recollapse)
4. and the cosmological red-shift.

Our first aim will be to make maximal use of the symmetries that simple cosmological models should have to find a simple ansatz for the metric. Our guiding principle will be . . .

15.2 Fundamental Observations I: The Cosmological Principle

At first, it may sound impossibly difficult to find solutions of the Einstein equations describing the universe as a whole. But: If one looks at the universe at large (very large) scales, in that process averaging over galaxies and even clusters of galaxies, then the situation simplifies a lot in several respects;

1. First of all, at those scales non-gravitational interactions can be completely ignored because they are either short-range (the nuclear forces) or compensate each other at large distances (electro-magnetism).
2. The earth, and our solar system, or even our galaxy, have no privileged position in the universe. This means that at large scales the universe should look the same from any point in the universe. Mathematically this means that there should be translational symmetries from any point of space to any other, in other words, space should be homogeneous.
3. Also, we assume that, at large scales, the universe looks the same in all directions. Thus there should be rotational symmetries and hence space should be isotropic.

It thus follows from our discussion in section 14, that the n-dimensional space (of course $n = 3$ for us) has n translational and $n(n-1)/2$ rotational Killing vectors, i.e. that the spatial metric is *maximally symmetric*. For $n = 3$, we will thus have six Killing vectors, two more than for the Schwarzschild metric, and the ansatz for the metric will simplify accordingly.

Note that since we know from observation that the universe expands, we do not require a maximally symmetric space-time as this would imply that there is also a timelike Killing vector and the resulting model for the universe would be static.

What simplifies life considerably is the fact that, as we have seen, there are only three species of maximally symmetric spaces (for any n), namely flat space $\mathbb{R}^n$, the sphere S^n, and its negatively curved counterpart, the n-dimensional pseudosphere or hyperboloid we will call H^n.

Thus, for a space-time metric with maximally symmetric spacelike 'slices', the only unknown is the time-dependence of the overall size of the metric. More concretely, the metric can be chosen to be

$$ds^2 = -dt^2 + a^2(t)(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2) \quad , \tag{15.1}$$

where $k = 0, \pm 1$ corresponds to the three possibilities mentioned above. Thus the metric contains only one unknown function, the 'radius' or cosmic scale factor $a(t)$. This function will be determined by the Einstein equations via the matter content of the universe (we will of course be dealing with a non-vanishing energy-momentum tensor) and the equation of state for the matter.

15.3 Fundamental Observations II: Olbers' Paradox

One paradox, popularised by Olbers (1826) but noticed before by others is the following. He asked the seemingly innocuous question "Why is the sky dark at night?". According to his calculation, reproduced below, the sky should instead be infinitely bright.

The simplest assumption one could make in cosmology (prior to the discovery of the Hubble expansion) is that the universe is static, infinite and homogeneously filled with stars. In fact, this is probably the naive picture one has in mind when looking at the stars at night, and certainly for a long time astronomers had no reason to believe otherwise.

However, these simple assumptions immediately lead to a paradox, namely the conclusion that the night-sky should be infinitely bright (or at least very bright) whereas, as we know, the sky is actually quite dark at night. This is a nice example of how very simple observations can actually tell us something deep about nature (in this case, the nature of the universe). The argument runs as follows.

1. Assume that there is a star of brightness (luminosity) L at distance r. Then, since the star sends out light into all directions, the apparent luminosity A (neglecting absorption) will be

$$A(r) = L/4\pi r^2 \quad . \tag{15.2}$$

2. If the number density ν of stars is constant, then the number of stars at distances between r and $r + dr$ is

$$dN(r) = 4\pi\nu r^2 dr \quad . \tag{15.3}$$

   Hence the total energy density due to the radiation of all the stars is

$$E = \int_0^\infty A(r)dN(r) = L\nu\int_0^\infty dr = \infty \quad . \tag{15.4}$$

3. Therefore the sky should be infinitely bright.

Now what is one to make of this? Clearly some of the assumptions in the above are much too naive. The way out suggested by Olbers is to take into account absorption effects and to postulate some absorbing interstellar medium. But this is also too naive because in an eternal universe we should now be in a stage of thermal equilibrium. Hence the postulated interstellar medium should emit as much energy as it absorbs, so this will not reduce the radiant energy density either.

Of course, the stars themselves are not transparent, so they could block out light completely from distant sources. But if this is to rescue the situation, one would need to postulate so many stars that every line of sight ends on a star, but then the night sky would be bright (though not infinitely bright) and not dark.

Modern cosmological models can resolve this problem in a variety of ways. For instance, the universe could be static but finite (there are such solutions, but this is nevertheless an unlikely scenario) or the universe is not eternal since there was a 'Big Bang' (and this is a more likely scenario).

15.4 Fundamental Observations III: The Hubble(-Lemaître) Expansion

We have already discussed one of the fundamental inputs of simple cosmological models, namely the cosmological principle. This led us to consider space-times with maximally-symmetric spacelike slices. One of the few other things that is definitely known about the universe, and that tells us something about the time-dependence of the universe, is that it expands or, at least that it appears to be expanding.

In fact, in the 1920's and 1930's, the astronomer Edwin Hubble made a remarkable discovery regarding the motion of galaxies. He found that light from distant galaxies is systematically red-shifted (increased in wave-length λ), the increase being proportional to the distance d of the galaxy,

$$z := \frac{\Delta\lambda}{\lambda} \propto d \ . \tag{15.5}$$

Hubble interpreted this red-shift as due to a Doppler effect and therefore ascribed a *recessional velocity* $v = cz$ to the galaxy. While, as we will see, this pure Doppler shift explanation is not tenable or at least not always the most useful way of phrasing things, the terminology has stuck, and Hubble's law can be written in the form

$$v = Hd \ , \tag{15.6}$$

where H is *Hubble's constant*. To set the historical record straight: credit for this fundamental discovery should perhaps (also) go to G. Lemaître.[7]

[7] See e.g. M. Way, H. Nussbaumer, *The linear redshift-distance relationship: Lematre beats Hubble by two years*, `arXiv:1104.3031v1 [physics.hist-ph]`; J.-P. Luminet, *Editorial note to "The beginning of the world from the point of view of quantum theory"*, `arXiv:1105.6271v1 [physics.hist-ph]`.

We will see later that in most cosmological models H is actually a function of time, so the H in the above equation should then be interpreted as the value H_0 of H today.

Actually, not only in cosmological toy-models but also in experiments, H is a function of time, with estimates fluctuating rather wildly over the years. It is one of the main goals of observational cosmology to determine H as precisely as possible, and the main problem here is naturally a precise determination of the distances of distant galaxies.

Galactic distances are frequently measured in *mega-parsecs* (Mpc). A parsec is the distance from which a star subtends an angle of 2 arc-seconds at the two diametrically opposite ends of the earth's orbit. This unit arose because of the old trigonometric method of measuring stellar distances (a triangle is determined by the length of one side and the two adjacent angles). 1 parsec is approximately 3×10^{18} cm, a little over 3 light-years. The Hubble constant is therefore often expressed in units of km s^{-1} (Mpc)$^{-1}$. The best currently available estimates point to a value of H_0 in the range (using a standard parametrisation)

$$\begin{aligned} H_0 &= 100h\text{km/s/Mpc} \ , \\ h &= 0.71 \pm 0.06 \ . \end{aligned} \tag{15.7}$$

We will usually prefer to express it just in terms of inverse units of time. The above result leads to an order of magnitude range of

$$H_0^{-1} \approx 10^{10} \text{ years} \tag{15.8}$$

(whereas Hubble's original estimate was more in the 10^9 year range).

15.5 Mathematical Model: the Robertson-Walker Metric

Having determined that the metric of a maximally symmetric space is of the simple form (14.27), we can now deduce that a space-time metric satisfying the Cosmological Principle can be chosen to be of the form (15.1),

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \ . \tag{15.9}$$

Here we have used the fact that (as in the ansatz for a spherically symmetric metric) non-trivial g_{tt} and g_{tr} can be removed by a coordinate transformation. This metric is known as the *Friedmann-Robertson-Walker metric* or just the Robertson-Walker metric, and spatial coordinates in which the metric takes this form are called *comoving coordinates*, for reasons that will become apparent below.

The metric of the three-space at constant t is

$$g_{ij} = a^2(t)\tilde{g}_{ij} \ , \tag{15.10}$$

where $\tilde{g}_{ij}$ is the maximally symmetric spatial metric. Thus for $k = +1$, $a(t)$ directly gives the size (radius) of the universe. For $k = -1$, space is infinite, so no such interpretation is possible, but nevertheless $a(t)$ still sets the scale for the geometry of the universe, e.g. in the sense that the curvature scalar $R^{(3)}$ of the metric g_{ij} is related to the curvature scalar $\tilde{R}^{(3)}$ of $\tilde{g}_{ij}$ by

$$R^{(3)}(t) = \frac{1}{a^2(t)}\tilde{R}^{(3)} \quad . \tag{15.11}$$

Finally, for $k = 0$, three-space is flat and also infinite, but one could replace $\mathbb{R}^3$ by a three-torus T^3 (still maximally symmetric and flat but now compact) and then $a(t)$ would once again be related directly to the size of the universe at constant t. Anyway, in all cases, $a(t)$ plays the role of a (and is known as the) *cosmic scale factor*.

Note that the case $k = +1$ opened up for the very first time the possibility of considering, even conceiving, an unbounded but finite universe! These and other generalisations made possible by a general relativistic approach to cosmology are important as more naive (Newtonian) models of the universe immediately lead to paradoxes or contradictions (as we have seen e.g. in the discussion of Olbers' paradox in section 15.3).

Let us now look at geodesics. Note that, since $g_{tt} = -1$ is a constant, one has

$$\Gamma_{\mu tt} = \frac{1}{2}(2\partial_t g_{\mu t} - \partial_\mu g_{tt}) = 0 \quad . \tag{15.12}$$

Therefore the vector field ∂_t is geodesic, which can be expressed as the statement that

$$\nabla_t \partial_t := \Gamma^\mu_{tt}\partial_\mu = 0 \quad . \tag{15.13}$$

In simpler terms this means that the curves $\vec{x} = const.$,

$$\tau \to (\vec{x}(\tau), t(\tau)) = (\vec{x}_0, \tau) \tag{15.14}$$

are geodesics. This also follows from the considerations around (2.48) in section 2.

Hence, in this coordinate system, observers remaining at fixed values of the spatial coordinates are in free fall. In other words, the coordinate system is falling with them or *comoving*, and the proper time along such geodesics coincides with the coordinate time, $d\tau = dt$. It is these observers of constant $\vec{x}$ or constant (r, θ, ϕ) who all see the same isotropic universe at a given value of t.

This may sound a bit strange but a good way to visualise such a coordinate system is, as in Figure 20, as a mesh of coordinate lines drawn on a balloon that is being inflated or deflated (according to the behaviour of $a(t)$). Draw some dots on that balloon (that will eventually represent galaxies or clusters of galaxies). As the ballon is being inflated or deflated, the dots will move but the coordinate lines will move with them and the dots remain at fixed spatial coordinate values. Thus, as we now know, regardless of the behaviour of $a(t)$, these dots follow a geodesic, and we will thus think of galaxies in this description as being in free fall.

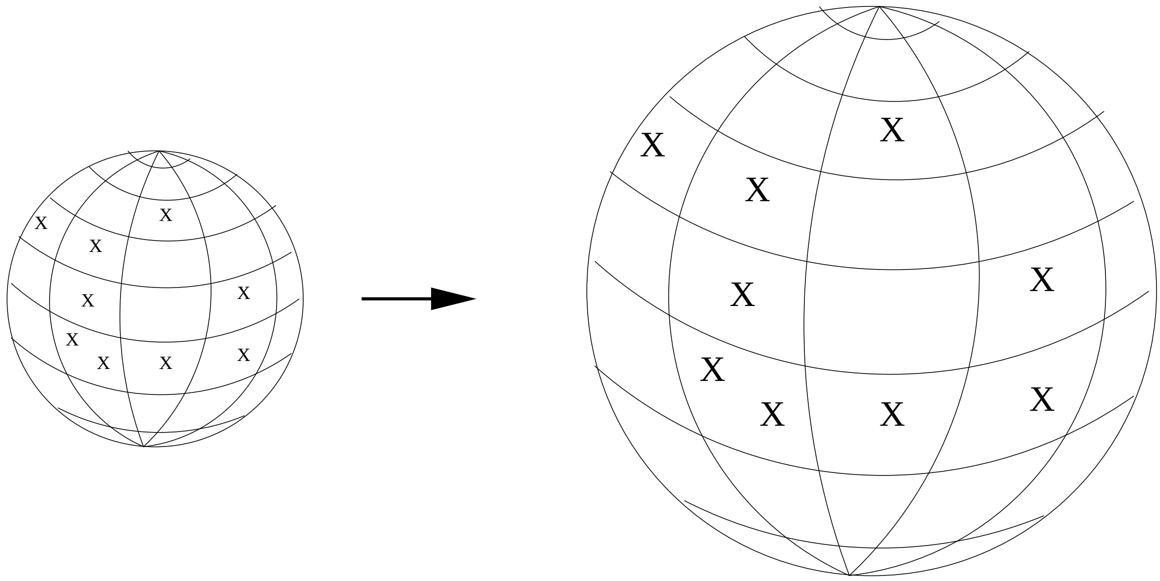

Figure 20: Illustration of a comoving coordinate system: Even though the sphere (universe) expands, the X's (galaxies) remain at the same spatial coordinates. These trajectories are geodesics and hence the X's (galaxies) can be considered to be in free fall. The figure also shows that it is the number density per unit coordinate volume that is conserved, *not* the density per unit proper volume.

Note that this immediately implies a crude distance - velocity relation reminiscent of Hubble's law. Namely, let us ego- or geocentrically place ourselves at the origin $r = 0$ (remember that because of maximal symmetry this point is as good as any other and in no way privileged). Consider another galaxy following the comoving geodesic at the fixed value $r = r_1$. Its "instantaneous" proper distance $R_1(t)$ at time t can be calculated from

$$dR = a(t)\frac{dr}{(1-kr^2)^{1/2}} \quad . \tag{15.15}$$

Choosing $k = 0$ for simplicity, this can be integrated to

$$R_1(t) = a(t)r_1 \quad . \tag{15.16}$$

It follows that

$$V_1(t) \equiv \frac{d}{dt}R_1(t) = \dot{a}(t)r_1 = H(t)R_1(t) \quad , \tag{15.17}$$

where we have introduced the *Hubble parameter*

$$H(t) = \frac{\dot{a}(t)}{a(t)} \quad . \tag{15.18}$$

The relation (15.17) clearly expresses something like Hubble's law $v = Hd$ (15.6): all objects run away from each other with velocities proportional to their distance. We will have much more to say about $H(t)$, and about the relation between distance and red-shift z, below.

Another advantage of the comoving coordinate system is that the six-parameter family of isometries just acts on the spatial part of the metric. Indeed, let $K^i\partial_i$ be a Killing vector of the maximally symmetric spatial metric. Then $K^i\partial_i$ is also a Killing vector of the Robertson-Walker metric. This would not be the case if one had e.g. made an x-dependent coordinate transformation of t or a t-dependent coordinate transformation of the x^i. In those cases there would of course still be six Killing vectors, but they would have a more complicated form.

15.6 * Area Measurements in a Robertson-Walker Metric and Number Counts

The aim of this and the subsequent sections is to learn as much as possible about the general properties of Robertson-Walker geometries (without using the Einstein equations) with the aim of looking for observational means of distinguishing e.g. among the models with $k = 0, \pm 1$.

To get a feeling for the geometry of the Schwarzschild metric, we studied the properties of areas and lengths in the Schwarzschild geometry. Length measurements are rather obvious in the Robertson-Walker geometry, so here we focus on the properties of areas.

We write the spatial part of the Robertson-Walker metric in polar coordinates as

$$ds^2 = a^2[d\psi^2 + f^2(\psi)d\Omega^2] \quad , \tag{15.19}$$

where $f(\psi) = \psi, \sin\psi, \sinh\psi$ for $k = 0, +1, -1$. Now the radius of a surface $\psi = \psi_0$ around the point $\psi = 0$ (or any other point, our space is isotropic and homogeneous) is given by

$$\rho = a\int_0^{\psi_0} d\psi = a\psi_0 \quad . \tag{15.20}$$

On the other hand, the area of this surface is determined by the induced metric $a^2f^2(\psi_0)d\Omega^2$ and is

$$A(\rho) = a^2f^2(\psi_0)\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta = 4\pi a^2 f^2(\rho/a) \quad . \tag{15.21}$$

For $k = 0$, this is just the standard behaviour

$$A(\rho) = 4\pi\rho^2 \quad , \tag{15.22}$$

but for $k = \pm 1$ the geometry looks quite different.

For $k = +1$, we have

$$A(\rho) = 4\pi a^2 \sin^2(\rho/a) \quad . \tag{15.23}$$

Thus the area reaches a maximum for $\rho = \pi a/2$ (or $\psi = \pi/2$), then *decreases* again for larger values of ρ and goes to zero as $\rho \to \pi a$. Already the maximal area, $A_{max} = 4\pi a^2$

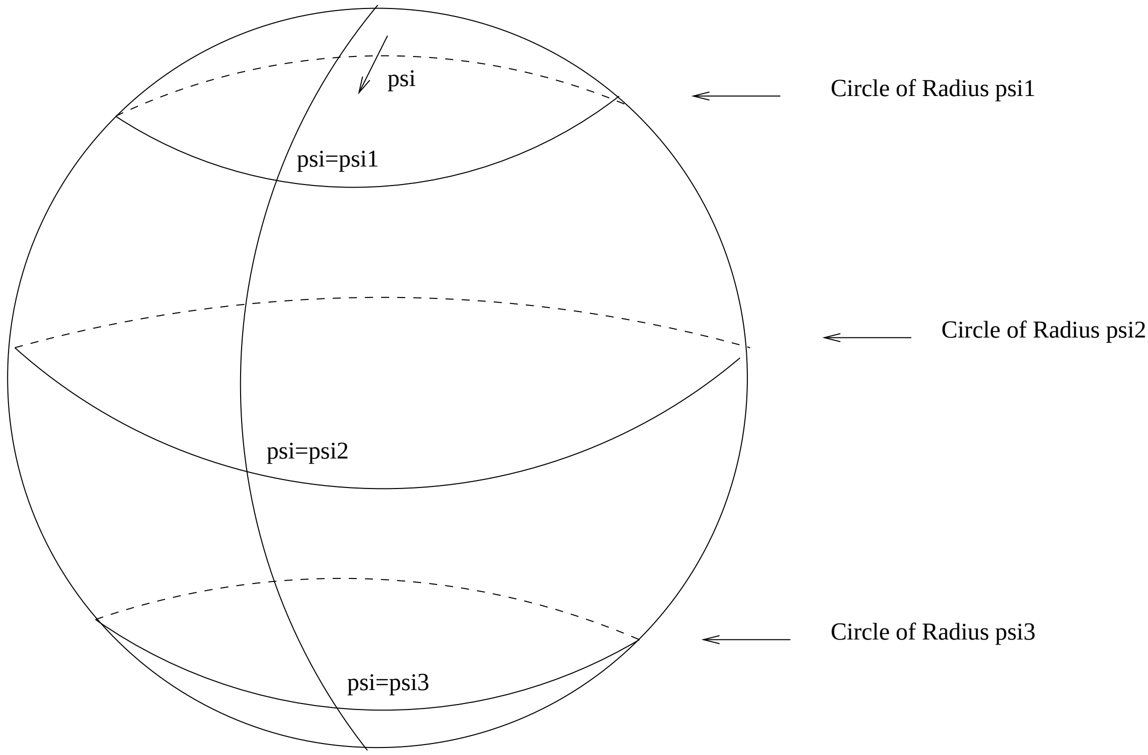

Figure 21: Visualisation of the $k = +1$ Robertson-Walker geometry via a two-sphere of unit radius: Circles of radius ψ, measured along the two-sphere, have an area which grows at first, reaches a maximum at $\psi = \pi/2$ and goes to zero when $\psi \to \pi$. E.g. the maximum value of the circumference, at $\psi = \pi/2$, namely 2π, is much smaller than the circumference of a circle with the same radius $\pi/2$ in a flat geometry, namely $2\pi \times \pi/2 = \pi^2$. Only for ψ very small does one approximately see a standard Euclidean geometry.

is much smaller than the area of a sphere of the same radius in Euclidean space, which would be $4\pi\rho^2 = \pi^3 a^2$.

This behaviour is best visualised by replacing the three-sphere by the two-sphere and looking at the circumference of circles as a function of their distance from the origin (see Figure 21).

For $k = -1$, we have

$$A(\rho) = 4\pi a^2 \sinh^2(\rho/a) \quad , \tag{15.24}$$

so in this case the area grows much more rapidly with the radius than in flat space.

In principle, this distinct behaviour of areas in the models with $k = 0, \pm 1$ might allow for an empirical determination of k. For instance, one might make the assumption that there is a homogeneous distribution of the number and brightness of galaxies, and one could try to determine observationally the number of galaxies as a function of their apparent luminosity. As in the discussion of Olbers' paradox, the radiation flux would be proportional to $F \propto 1/\rho^2$. In Euclidean space ($k = 0$), one would expect the number $N(F)$ of galaxies with flux greater than F, i.e. distances less than ρ to behave like ρ^3, so that the expected Euclidean behaviour would be

$$N(F) \propto F^{-3/2} \quad . \tag{15.25}$$

Any empirical departure from this behaviour could thus be an indication of a universe with $k \neq 0$, but clearly, to decide this, many other factors (red-shift, evolution of stars, etc.) have to be taken into account and so far it has been impossible to determine the value of k in this way.

15.7 The Cosmological Red-Shift

The most important information about the cosmic scale factor $a(t)$ comes from the observation of shifts in the frequency of light emitted by distant sources. To calculate the expected shift in a Robertson-Walker geometry, let us again place ourselves at the origin $r = 0$. We consider a radially travelling electro-magnetic wave (a light ray) and consider the equation $d\tau^2 = 0$ or

$$dt^2 = a^2(t)\frac{dr^2}{1-kr^2} \quad . \tag{15.26}$$

Let us assume that the wave leaves a galaxy located at $r = r_1$ at the time t_1. Then it will reach us at a time t_0 given by

$$f(r_1) = \int_{r_1}^{0} \frac{dr}{\sqrt{1-kr^2}} = \int_{t_1}^{t_0} \frac{dt}{a(t)} \quad . \tag{15.27}$$

As typical galaxies will have constant coordinates, $f(r_1)$ (which can of course be given explicitly, but this is not needed for the present analysis) is time-independent. If the next wave crest leaves the galaxy at r_1 at time $t_1 + \delta t_1$, it will arrive at a time $t_0 + \delta t_0$ determined by

$$f(r_1) = \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{a(t)} \quad . \tag{15.28}$$

Subtracting these two equations and making the (eminently reasonable) assumption that the cosmic scale factor $a(t)$ does not vary significantly over the period δt given by the frequency of light, we obtain

$$\frac{\delta t_0}{a(t_0)} = \frac{\delta t_1}{a(t_1)} \quad . \tag{15.29}$$

Indeed, say that $b(t)$ is the integral of $1/a(t)$. Then we have

$$b(t_0 + \delta t_0) - b(t_1 + \delta t_1) = b(t_0) - b(t_1) \quad , \tag{15.30}$$

and Taylor expanding to first order, we obtain

$$b'(t_0)\delta t_0 = b'(t_1)\delta t_1 \quad , \tag{15.31}$$

which is the same as (15.29). Therefore the observed frequency ν_0 is related to the emitted frequency ν_1 by

$$\frac{\nu_0}{\nu_1} = \frac{a(t_1)}{a(t_0)} \quad . \tag{15.32}$$

Astronomers like to express this in terms of the red-shift parameter (see the discussion of Hubble's law above)

$$z = \frac{\lambda_0 - \lambda_1}{\lambda_1} \quad , \tag{15.33}$$

which in view of the above result we can write as

$$z = \frac{a(t_0)}{a(t_1)} - 1 \quad . \tag{15.34}$$

Thus if the universe expands one has $z > 0$ and there is a red-shift while in a contracting universe with $a(t_0) < a(t_1)$ the light of distant glaxies would be blue-shifted.

A few remarks on this result:

1. This cosmological red-shift has nothing to do with the star's own gravitational field - that contribution to the red-shift is completely negligible compared to the effect of the cosmological red-shift.
2. Unlike the gravitational red-shift we discussed before, this cosmological red-shift is symmetric between receiver and emitter, i.e. light sent from the earth to the distant galaxy would likewise be red-shifted if we observe a red-shift of the distant galaxy.
3. This red-shift is a combined effect of gravitational and Doppler red-shifts and, without additional choices and input, it is not very meaningful to separate this into the two and/or to interpret this only in terms of, say, a Doppler shift. Nevertheless, as mentioned before, astronomers like to do just that, calling $v = zc$ the recessional velocity.[8]
4. Nowadays, astronomers tend to express the distance of a galaxy not in terms of light-years or megaparsecs, but directly in terms of the observed red-shift factor z, the conversion to distance then following from some version of Hubble's law.
5. The largest observed redshift of a galaxy is currently $z \approx 10$, corresponding to a distance of the order of 13 billion light-years, while the cosmic microwave background radiation, which originated just a couple of 100.000 years after the Big Bang, has $z > 1000$.

[8]For a lucid discussion of this issue, actually arguing in favour of the pure Doppler interpretation, see E. Bunn, D. Hogg, *The kinematic origin of the cosmological redshift*, `arXiv:0808.1081`.

15.8 The Red-Shift Distance Relation (Hubble's Law)

We have seen that there is a cosmological red-shift in Robertson-Walker geometries. Our aim will now be to see if and how these geometries are capable of explaining Hubble's law that the red-shift is approximately proportional to the distance and how the Hubble constant is related to the cosmic scale factor $a(t)$.

Reliable data for cosmological red-shifts as well as for distance measurements are only available for small values of z, and thus we will consider the case where $t_0 - t_1$ and r_1 are small, i.e. small at cosmic scales. First of all, this allows us to expand $a(t)$ in a Taylor series,

$$a(t) = a(t_0) + (t - t_0)\dot{a}(t_0) + \tfrac{1}{2}(t - t_0)^2\ddot{a}(t_0) + \ldots \tag{15.35}$$

Let us introduce the *Hubble parameter* $H(t)$ (which already made a brief appearance in section 15.5) and the *deceleration parameter* $q(t)$ by

$$\begin{aligned} H(t) &= \frac{\dot{a}(t)}{a(t)} \\ q(t) &= -\frac{a(t)\ddot{a}(t)}{\dot{a}(t)^2} \ , \end{aligned} \tag{15.36}$$

and denote their present day values by a subscript zero, i.e. $H_0 = H(t_0)$ and $q_0 = q(t_0)$. $H(t)$ measures the expansion velocity as a function of time while $q(t)$ measures whether the expansion velocity is increasing or decreasing. We will also denote $a_0 = a(t_0)$ and $a(t_1) = a_1$.

In terms of these parameters, the Taylor expansion can be written as

$$a(t) = a_0(1 + H_0(t - t_0) - \tfrac{1}{2}q_0 H_0^2 (t - t_0)^2 + \ldots) \ . \tag{15.37}$$

This gives us the red-shift parameter z as a power series in the time of flight, namely

$$\frac{1}{1+z} = \frac{a_1}{a_0} = 1 + (t_1 - t_0)H_0 - \tfrac{1}{2}q_0 H_0^2 (t_1 - t_0)^2 + \ldots \tag{15.38}$$

or

$$z = (t_0 - t_1)H_0 + (1 + \tfrac{1}{2}q_0)H_0^2 (t_0 - t_1)^2 + \ldots \tag{15.39}$$

For small $H_0(t_0 - t_1)$ this can be inverted,

$$t_0 - t_1 = \frac{1}{H_0}[z - (1 + \tfrac{1}{2}q_0)z^2 + \ldots] \ . \tag{15.40}$$

We can also use (15.27) to express $(t_0 - t_1)$ in terms of r_1. On the one hand we have

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = r_1 + \mathcal{O}(r_1^3) \ , \tag{15.41}$$

while expanding $a(t)$ in the denominator we get

$$\begin{aligned}\int_{t_1}^{t_0}\frac{dt}{a(t)} &= \frac{1}{a_0}\int_{t_1}^{t_0}\frac{dt}{(1+(t-t_0)H_0+\ldots)}\\ &= \frac{1}{a_0}\int_{t_1}^{t_0} dt\,[1+(t_0-t)H_0+\ldots]\\ &= \frac{1}{a_0}[(t_0-t_1)+t_0(t_0-t_1)H_0-\tfrac{1}{2}(t_0^2-t_1^2)H_0+\ldots]\\ &= \frac{1}{a_0}[(t_0-t_1)+\tfrac{1}{2}(t_0-t_1)^2H_0+\ldots]\ . \end{aligned} \tag{15.42}$$

Therefore we obtain

$$r_1 = \frac{1}{a_0}[(t_0-t_1)+\tfrac{1}{2}(t_0-t_1)^2H_0+\ldots]\ . \tag{15.43}$$

Using (15.40), we obtain

$$r_1 = \frac{1}{a_0H_0}[z-\tfrac{1}{2}(1+q_0)z^2+\ldots]\ . \tag{15.44}$$

This clearly indicates to first order a linear dependence of the red-shift on the distance of the galaxy and identifies H_0, the present day value of the Hubble parameter, as being at least proportional to the Hubble constant introduced in (15.6).

However, this is not yet a very useful way of expressing Hubble's law. First of all, the distance a_0r_1 that appears in this expression is not the proper distance (unless $k=0$), but is at least equal to it in our approximation. Note that a_0r_1 is the present distance to the galaxy, not the distance at the time the light was emitted.

However, even proper distance is not directly measurable and thus, to compare this formula with experiment, one needs to relate r_1 to the measures of distance used by astronomers. One practical way of doing this is the so-called *luminosity distance* d_L. If for some reasons one knows the absolute luminosity of a distant star (for instance because it shows a certain characteristic behaviour known from other stars nearby whose distances can be measured by direct means - such objects are known as *standard candles*), then one can compare this absolute luminosity L with the apparent luminosity A. Then one can define the luminosity distance d_L by (cf. (15.2))

$$d_L^2 = \frac{L}{4\pi A}\ . \tag{15.45}$$

We thus need to relate d_L to the coordinate distance r_1. The key relation is

$$\frac{A}{L} = \frac{1}{4\pi a_0^2r_1^2}\frac{1}{1+z}\frac{a_1}{a_0} = \frac{1}{4\pi a_0^2r_1^2(1+z)^2}\ . \tag{15.46}$$

Here the first factor arises from dividing by the area of the sphere at distance a_0r_1 and would be the only term in a flat geometry (see the discssion of Olbers' paradox). In a Robertson-Walker geometry, however, the photon flux will be diluted. The second factor is due to the fact that each individual photon is being red-shifted. And the third factor (identical to the second) is due to the fact that as a consequence of the expansion of the universe, photons emitted a time δt apart will be measured a time $(1+z)\delta t$ apart.

Hence the relation between r_1 and d_L is

$$d_L = (L/4\pi A)^{1/2} = r_1 a(t_0)(1+z) \ . \tag{15.47}$$

Intuitively, the fact that for z positive d_L is larger than the actual (proper) distance of the galaxy can be understood by noting that the gravitational red-shift makes an object look darker (further away) than it actually is.

This can be inserted into (15.44) to give an expression for the red-shift in terms of d_L, *Hubble's law*

$$d_L = H_0^{-1}[z + \tfrac{1}{2}(1-q_0)z^2 + \ldots] \ . \tag{15.48}$$

The program would then be to collect as much astronomical information as possible on the relation between d_L and z in order to determine the parameters q_0 and H_0.

16 Cosmology II: Basics of Friedmann-Robertson-Walker Cosmology

So far, we have only used the kinematical framework provided by the Robertson-Walker metrics and we never used the Einstein equations. The benefit of this is that it allows one to deduce relations betweens observed quantities and assumptions about the universe which are valid even if the Einstein equations are not entirely correct, perhaps because of higher derivative or other quantum corrections in the early universe.

Now, on the other hand we will have to be more specific, specify the matter content and solve the Einstein equations for $a(t)$. We will see that a lot about the solutions of the Einstein equations can already be deduced from a purely qualitative analysis of these equations, without having to resort to explicit solutions (Chapter 17). Exact solutions will then be the subject of Chapter 18.

16.1 The Ricci Tensor of the Robertson-Walker Metric

Of course, the first thing we need to discuss solutions of the Einstein equations is the Ricci tensor of the Robertson-Walker (RW) metric. Since we already know the curvature tensor of the maximally symmetric spatial metric entering the RW metric (and its contractions), this is not difficult.

1. First of all, we write the RW metric as

$$ds^2 = -dt^2 + a^2(t)\tilde{g}_{ij}dx^i dx^j \quad . \tag{16.1}$$

   From now on, all objects with a tilde, $\tilde{}$, will refer to three-dimensional quantities calculated with the metric $\tilde{g}_{ij}$.

2. One can then calculate the Christoffel symbols in terms of $a(t)$ and $\tilde{\Gamma}^i{}_{jk}$. The non-vanishing components are (we had already established that $\Gamma^\mu{}_{00} = 0$)

$$\begin{aligned} \Gamma^i{}_{jk} &= \tilde{\Gamma}^i{}_{jk} \\ \Gamma^i{}_{j0} &= \frac{\dot{a}}{a}\delta^i_j \\ \Gamma^0{}_{ij} &= \dot{a}a\tilde{g}_{ij} \end{aligned} \tag{16.2}$$

3. The relevant components of the Riemann tensor are

$$\begin{aligned} R^i{}_{0j0} &= -\frac{\ddot{a}}{a}\delta^i_j \\ R^0{}_{i0j} &= a\ddot{a}\tilde{g}_{ij} \\ R^k{}_{ikj} &= \tilde{R}_{ij} + 2\dot{a}^2\tilde{g}_{ij} \quad . \end{aligned} \tag{16.3}$$

4. Now we can use $\tilde{R}_{ij} = 2k\tilde{g}_{ij}$ (a consequence of the maximal symmetry of $\tilde{g}_{ij}$) to calculate $R_{\mu\nu}$. The non-zero components are

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a} \\ R_{ij} &= (a\ddot{a} + 2\dot{a}^2 + 2k)\tilde{g}_{ij} \\ &= (\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2k}{a^2})g_{ij} \quad . \end{aligned} \tag{16.4}$$

5. Thus the Ricci scalar is

$$R = \frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + k) \quad , \tag{16.5}$$

and

6. the Einstein tensor has the components

$$\begin{aligned} G_{00} &= 3(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}) \\ G_{0i} &= 0 \\ G_{ij} &= -(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2})g_{ij} \quad . \end{aligned} \tag{16.6}$$

16.2 The Matter Content: A Perfect Fluid

Next we need to specify the matter content. On physical grounds one might like to argue that in the approximation underlying the cosmological principle galaxies (or clusters) should be treated as non-interacting particles or a perfect fluid. As it turns out, we do not need to do this as the symmetries of the metric fix the energy-momentum tensor to be that of a perfect fluid anyway.

Below, I will give a formal argument for this using Killing vectors. But informally we can already deduce this from the structure of the Einstein tensor obtained above. Comparing (16.6) with the Einstein equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, we deduce that the Einstein equations can only have a solution with a Robertson-Walker metric if the energy-momentum tensor is of the form

$$\begin{aligned} T_{00} &= \rho(t) \\ T_{0i} &= 0 \\ T_{ij} &= p(t)g_{ij} \quad , \end{aligned} \tag{16.7}$$

where $p(t)$ and $\rho(t)$ are some functions of time.

Here is the formal argument. It is of course a consequence of the Einstein equations that any symmetries of the Ricci (or Einstein) tensor also have to be symmetries of the energy-momentum tensor. Now we know that the metric $\tilde{g}_{ij}$ has six Killing vectors $K^{(a)}$ and that (in the comoving coordinate system) these are also Killing vectors of the RW metric,

$$L_{K^{(a)}}\tilde{g}_{ij} = 0 \;\;\Rightarrow\;\; L_{K^{(a)}}g_{\mu\nu} = 0 \quad . \tag{16.8}$$

Therefore also the Ricci and Einstein tensors have these symmetries,

$$L_{K^{(a)}} g_{\mu\nu} = 0 \;\; \Rightarrow \;\; L_{K^{(a)}} R_{\mu\nu} = 0 \;\; , \;\; L_{K^{(a)}} G_{\mu\nu} = 0 \;\; . \tag{16.9}$$

Hence the Einstein equations imply that $T_{\mu\nu}$ should have these symmetries,

$$L_{K^{(a)}} G_{\mu\nu} = 0 \;\; \Rightarrow \;\; L_{K^{(a)}} T_{\mu\nu} = 0 \;\; . \tag{16.10}$$

Moreover, since the $L_{K^{(a)}}$ act like three-dimensional coordinate transformations, in order to see what these conditions mean we can make a $(3+1)$-decomposition of the energy-momentum tensor. From the three-dimensional point of view, T_{00} transforms like a scalar under coordinate transformations (and Lie derivatives), T_{0i} like a vector, and T_{ij} like a symmetric tensor. Thus we need to determine what are the three-dimensional scalars, vectors and symmetric tensors that are invariant under the full six-parameter group of the three-dimensional isometries.

For scalars ϕ we thus require (calling K now any one of the Killing vectors of $\tilde{g}_{ij}$),

$$L_K \phi = K^i \partial_i \phi = 0 \;\; . \tag{16.11}$$

But since $K^i(x)$ can take any value in a maximally symmetric space (homogeneity), this implies that ϕ has to be constant (as a function on the three-dimensional space) and therefore T_{00} can only be a function of time,

$$T_{00} = \rho(t) \;\; . \tag{16.12}$$

For vectors, it is almost obvious that no invariant vectors can exist because any vector would single out a particular direction and therefore spoil isotropy. The formal argument (as a warm up for the argument for tensors) is the following. We have

$$L_K V^i = K^j \tilde{\nabla}_j V^i + V^j \tilde{\nabla}_j K^i \;\; . \tag{16.13}$$

We now choose the Killing vectors such that $K^i(x) = 0$ but $\tilde{\nabla}_i K_j \equiv K_{ij}$ is an arbitrary antisymmetric matrix. Then the first term disappears and we have

$$L_K V^i = 0 \;\; \Rightarrow \;\; K_{ij} V^j = 0 \;\; . \tag{16.14}$$

To make the antisymmetry manifest, we rewrite this as

$$K_{ij} V^j = K_{kj} \delta^k_i V^j = 0 \;\; . \tag{16.15}$$

If this is to hold for all antisymmetric matrices, we must have

$$\delta^k_i V^j = \delta^j_i V^k \;\; , \tag{16.16}$$

and by contraction one obtains $nV^j = V^j$, and hence $V_j = 0$. Therefore, as expected, there is no invariant vector field and

$$T_{0i} = 0 \;\; . \tag{16.17}$$

We now come to symmetric tensors. Once again we choose our Killing vectors to vanish at a given point x and such that K_{ij} is an arbitrary antisymmetric matrix. Then the condition

$$L_K T_{ij} = K^k \tilde{\nabla}_k T_{ij} + \tilde{\nabla}_i K^k T_{kj} + \tilde{\nabla}_j K^k T_{ik} = 0 \tag{16.18}$$

reduces to

$$K_{mn}(\tilde{g}^{mk}\delta^n_i T_{kj} + \tilde{g}^{mk}\delta^n_j T_{ik}) = 0 \quad . \tag{16.19}$$

If this is to hold for all antisymmetric matrices K_{mn}, the antisymmetric part of the term in brackets must be zero or, in other words, it must be symmetric in the indices m and n, i.e.

$$\tilde{g}^{mk}\delta^n_i T_{kj} + \tilde{g}^{mk}\delta^n_j T_{ik} = \tilde{g}^{nk}\delta^m_i T_{kj} + \tilde{g}^{nk}\delta^m_j T_{ik} \quad . \tag{16.20}$$

Contracting over the indices n and i, one obtains

$$n\tilde{g}^{mk}T_{kj} + \tilde{g}^{mk}T_{jk} = \tilde{g}^{mk}T_{kj} + \delta^m_j T^k_{\ k} \quad . \tag{16.21}$$

Therefore

$$T_{ij} = \frac{\tilde{g}_{ij}}{n} T^k_{\ k} \quad . \tag{16.22}$$

But we already know that the scalar $T^k_{\ k}$ has to be a constant. Thus we conclude that the only invariant tensor is the metric itself, and therefore the T_{ij}-components of the energy-momentum tensor can only be a function of t times $\tilde{g}_{ij}$. Writing this function as $p(t)a^2(t)$, we arrive at

$$T_{ij} = p(t) g_{ij} \quad . \tag{16.23}$$

We thus see that the energy-momentum tensor is determined by two functions, $\rho(t)$ and $p(t)$.

A covariant way of writing this tensor is as

$$T_{\mu\nu} = (p + \rho) u_\mu u_\nu + p g_{\mu\nu} \quad , \tag{16.24}$$

where $u^\mu = (1, 0, 0, 0)$ in a comoving coordinate system. This is precisely the energy-momentum tensor of a *perfect fluid*. u_μ is known as the *velocity field* of the fluid, and the comoving coordinates are those with respect to which the fluid is at rest. ρ is the energy-density of the perfect fluid and p is the pressure.

In general, this matter content has to be supplemented by an equation of state. This is usually assumed to be that of a *barytropic fluid*, i.e. one whose pressure depends only on its density, $p = p(\rho)$. The most useful toy-models of cosmological fluids arise from considering a linear relationship between p and ρ, of the type

$$p = w\rho \quad , \tag{16.25}$$

where w is known as the equation of state parameter. Occasionally also more exotic equations of state are considered.

For non-interacting particles, there is no pressure, $p = 0$, i.e. $w = 0$, and such matter is usually referred to as *dust*.

The trace of the energy-momentum tensor is

$$T^\mu_{\ \mu} = -\rho + 3p \quad . \tag{16.26}$$

For radiation, for example, the energy-momentum tensor is (like that of Maxwell theory) traceless, and hence *radiation* has the equation of state

$$p = \rho/3 \quad , \tag{16.27}$$

and thus $w = 1/3$.

For physical (gravitating instead of anti-gravitating) matter one usually requires $\rho > 0$ (positive energy) and either $p > 0$, corresponding to $w > 0$ or, at least, $\rho + 3p > 0$, corresponding to the weaker condition $w > -1/3$.

A cosmological constant Λ, on the other hand, corresponds, as we will see, to a matter contribution with $w = -1$ and thus violates either $\rho > 0$ or $\rho + 3p > 0$.

16.3 Conservation Laws

The same arguments as above show that a current J^μ in a Robertson-Walker metric has to be of the form $J^\mu = (n(t), 0, 0, 0)$ in comoving coordinates, or

$$J^\mu = n(t)u^\mu \tag{16.28}$$

in covariant form. Here $n(t)$ could be a number density like a galaxy number density. It gives the number density per unit proper volume. The conservation law $\nabla_\mu J^\mu = 0$ is equivalent to

$$\nabla_\mu J^\mu = 0 \quad \Leftrightarrow \quad \partial_t(\sqrt{g}n(t)) = 0 \ . \tag{16.29}$$

Thus we see that $n(t)$ is not constant, but the number density per unit coordinate volume is (as we had already anticipated in the picture of the balloon, Figure 20). For a RW metric, the time-dependent part of $\sqrt{g}$ is $a(t)^3$, and thus the conservation law says

$$n(t)a(t)^3 = const. \tag{16.30}$$

Let us now turn to the conservation laws associated with the energy-momentum tensor,

$$\nabla_\mu T^{\mu\nu} = 0 \ . \tag{16.31}$$

The spatial components of this conservation law,

$$\nabla_\mu T^{\mu i} = 0 \ , \tag{16.32}$$

turn out to be identically satisfied, by virtue of the fact that the u^μ are geodesic and that the functions ρ and p are only functions of time. This could hardly be otherwise because $\nabla_\mu T^{\mu i}$ would have to be an invariant vector, and we know that there are none (nevertheless it is instructive to check this explicitly).

The only interesting conservation law is thus the zero-component

$$\nabla_\mu T^{\mu 0} = \partial_\mu T^{\mu 0} + \Gamma^\mu_{\ \mu\nu} T^{\nu 0} + \Gamma^0_{\ \mu\nu} T^{\mu\nu} = 0 \ , \tag{16.33}$$

which for a perfect fluid becomes

$$\partial_t \rho(t) + \Gamma^\mu_{\ \mu 0}\rho(t) + \Gamma^0_{\ 00}\rho(t) + \Gamma^0_{\ ij} T^{ij} = 0 \ . \tag{16.34}$$

Inserting the explicit expressions (16.2) for the Christoffel symbols, this becomes

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a} \quad . \tag{16.35}$$

For instance, when the pressure of the cosmic matter is negligible, like in the universe today, and we can treat the galaxies (without disrespect) as dust, then one has

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a} \quad , \tag{16.36}$$

and this equation can trivially be integrated to

$$\rho(t)a(t)^3 = const. \tag{16.37}$$

On the other hand, if the universe is dominated by, say, radiation, then one has the equation of state $p = \rho/3$, and the conservation equation reduces to

$$\frac{\dot{\rho}}{\rho} = -4\frac{\dot{a}}{a} \quad , \tag{16.38}$$

and therefore

$$\rho(t)a(t)^4 = const. \tag{16.39}$$

The reason why the energy density of photons decreases faster with $a(t)$ than that of dust is of course ... the red-shift.

More generally, for matter with equation of state parameter w one finds

$$\rho(t)a(t)^{3(1+w)} = const. \tag{16.40}$$

In particular, for $w = -1$, ρ is constant and corresponds, as we will see more explicitly below, to a cosmological constant Λ. This can also be deduced by comparing the covariant form (16.24) of the energy-momentum tensor with the contribution of a cosmological constant to the Einstein equations, which is proportional to $g_{\mu\nu}$. By (16.24) this implies $p = -\rho$.

16.4 The Einstein and Friedmann Equations

After these preliminaries, we are now prepared to tackle the Einstein equations. We allow for the presence of a cosmological constant and thus consider the equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \quad . \tag{16.41}$$

It will be convenient to rewrite these equations in the form

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T^{\lambda}_{\ \lambda}) + \Lambda g_{\mu\nu} \quad . \tag{16.42}$$

Because of isotropy, there are only two independent equations, namely the 00-component and any one of the non-zero ij-components. Using (16.4), we find

$$\begin{aligned} -3\frac{\ddot{a}}{a} &= 4\pi G(\rho + 3p) - \Lambda \\ \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2} &= 4\pi G(\rho - p) + \Lambda \ . \end{aligned} \tag{16.43}$$

We supplement this by the conservation equation

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a} \ . \tag{16.44}$$

Using the first equation to eliminate $\ddot{a}$ from the second, one obtains the set of equations

$$\boxed{\begin{array}{lrcl} (F1) & \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} &=& \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} \\ (F2) & -3\frac{\ddot{a}}{a} &=& 4\pi G(\rho + 3p) - \Lambda \\ (F3) & \dot{\rho} &=& -3(\rho + p)\frac{\dot{a}}{a} \end{array}} \tag{16.45}$$

Together, these are known as the *Friedmann equations*. They govern every aspect of Friedmann-Robertson-Walker cosmology. From now on I will simply refer to them as equations (F1), (F2), (F3) respectively. In terms of the Hubble parameter $H(t)$ and the deceleration parameter $q(t)$, these equations can also be written as

$$\begin{array}{lrcl} (F1') & H^2 &=& \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \\ (F2') & q &=& \frac{1}{3H^2}[4\pi G(\rho + 3p) - \Lambda] \\ (F3') & \frac{d}{dt}(\rho a^3) &=& -3Hpa^3 \ . \end{array}$$

Note that because of the Bianchi identities, the Einstein equations and the conservation equations should not be independent, and indeed they are not.

It is easy to see that (F1) and (F3) imply the second order equation (F2) so that, a pleasant simplification, in practice one only has to deal with the two first order equations (F1) and (F3). Sometimes, however, (F2) is easier to solve than (F1), because it is linear in $\ddot{a}(t)$, and then (F1) is just used to fix one constant of integration.

It is also equally to see that (F1) and (F2) imply (F3), i.e. that the gravity equations of motion imply the matter equations of motion, a general and fundamental feature of general relativity.

Finally, formally (F2) and (F3) also imply (F1), with k (which only appears in (F1)) arising as an integration constant.

17 Cosmology III: Qualitative Analysis

A lot can be deduced about the solutions of the Friedmann equations, i.e. the evolution of the universe in the Friedmann-Robertson-Walker cosmologies, without solving the equations directly and even without specifying a precise equation of state, i.e. a relation between p and ρ. In the following we will, in turn, discuss the Big Bang, the age of the universe, and its long term behaviour, from this qualitative point of view. I will then introduce the notions of critical density and density parameters, and discuss the structure of the universe, as we understand it today, in these terms.

17.1 The Big Bang

One amazing thing about the FRW models is that all of them (provided that the matter content is reasonably physical) predict an initial singularity, commonly known as a Big Bang. This is very easy to see.

(F2) shows that, as long as the right-hand-side is positive, one has $q > 0$, i.e. $\ddot{a} < 0$ so that the universe is decelerating due to gravitational attraction. This is for instance the case when there is no cosmological constant and $\rho + 3p$ is positive (and this is the case for all physical matter). It is also true for a negative cosmological constant (its negative energy density being outweighed by 3 times its positive pressure). It need not be true, however, in the presence of a positive cosmological constant which provides an accelerating contribution to the expansion of the universe. We will, for the time being, continue with the assumption that Λ is zero or, at least, non-positive, even though, as we will discuss later, recent evidence (strongly) suggests the presence of a non-negligible positive cosmological constant in our universe today.

Since $a > 0$ by definition, $\dot{a}(t_0) > 0$ because we observe a red-shift, and $\ddot{a} < 0$ because $\rho + 3p > 0$, it follows that there cannot have been a turning point in the past and $a(t)$ must be concave downwards. Therefore $a(t)$ must have reached $a = 0$ at some *finite* time in the past. We will call this time $t = 0$, $a(0) = 0$.

As ρa^4 is constant for radiation (an appropriate description of earlier periods of the universe), this shows that the energy density grows like $1/a^4$ as $a \to 0$ so this leads to quite a singular situation.

Once again, as in our discussion of black holes, it is natural to wonder at this point if the singularities predicted by General Relativity in the case of cosmological models are generic or only artefacts of the highly symmetric situations we were considering. And again there are singularity theorems applicable to these situations which state that, under reasonable assumptions about the matter content, singularities will occur independently of assumptions about symmetries.

17.2 The Age of the Universe

With the normalisation $a(0) = 0$, it is fair to call t_0 the age of the universe. If $\ddot{a}$ had been zero in the past for all $t \leq t_0$, then we would have

$$\ddot{a} = 0 \quad \Rightarrow \quad a(t) = a_0 t / t_0 \quad , \tag{17.1}$$

and

$$\dot{a}(t) = a_0/t_0 = \dot{a}_0 \quad . \tag{17.2}$$

This would determine the age of the universe to be

$$t_0 = \frac{a_0}{\dot{a}_0} = H_0^{-1} \quad , \tag{17.3}$$

where H_0^{-1} is the *Hubble time*. However, provided that $\ddot{a} < 0$ for $t \leq t_0$ (as discussed above, this holds under suitable conditions on the matter content - which may or may not be realised in our universe), the actual age of the universe must be smaller than this,

$$\ddot{a} < 0 \quad \Rightarrow \quad t_0 < H_0^{-1} \quad . \tag{17.4}$$

Thus the Hubble time sets an upper bound on the age of the universe. See Figure 22 for an illustration of this.

17.3 Long Term Behaviour

Let us now try to take a look into the future of the universe. Again we will see that it is remarkably simple to extract relevant information from the Friedmann equations without ever having to solve an equation.

We will assume that $\Lambda = 0$ and that we are dealing with physical matter with $w > -1/3$. The Friedmann equation (F1) can be written as

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - k \quad . \tag{17.5}$$

The left-hand-side is manifestly non-negative. Let us see what this tells us about the right-hand-side. Focus on the first term $\sim \rho a^2$. This term is strictly positive and, according to (16.40), behaves as

$$\rho a^2 \sim a^{-3(1+w)+2} = a^{-(1+3w)} \quad . \tag{17.6}$$

Thus for physical matter the exponent is negative, so that if and when the cosmic scale factor $a(t)$ goes to infinity, one has

$$\lim_{a\to\infty} \rho a^2 = 0 \quad . \tag{17.7}$$

Now let us look at the second term on the right-hand-side of (17.5), and analyse the 3 choices for k. For $k = -1$ or $k = 0$, the right hand side of (17.5) is strictly positive. Therefore $\dot{a}$ is never zero and since $\dot{a}_0 > 0$, we must have

$$\dot{a}(t) > 0 \ \forall\, t \ . \tag{17.8}$$

Thus we can immediately conclude that open and flat universes must expand forever, i.e. they are open in space *and* time.

By taking into account (17.7), we can even be somewhat more precise about the long term behaviour. For $k = 0$, we learn that

$$k = 0: \qquad \lim_{a\to\infty} \dot{a}^2 = 0 \ . \tag{17.9}$$

Thus the universe keeps expanding but more and more slowly as time goes on. By the same reasoning we see that for $k = -1$ we have

$$k = -1: \qquad \lim_{a\to\infty} \dot{a}^2 = 1 \ . \tag{17.10}$$

Thus the universe keeps expanding, reaching a constant limiting velocity.

For $k = +1$, validity of (17.7) would lead us to conclude that $\dot{a}^2 \to -1$, but this is obviously a contradiction. Therefore we learn that the $k = +1$ universes never reach $a \to \infty$ and that there is therefore a maximal radius a_{max}. This maximal radius occurs for $\dot{a} = 0$ and therefore

$$k = +1: \qquad a_{max}^2 = \frac{3}{8\pi G\rho} \ . \tag{17.11}$$

Note that intuitively this makes sense. For larger ρ or larger G the gravitational attraction is stronger, and therefore the maximal radius of the universe will be smaller. Since we have $\ddot{a} < 0$ also at a_{max}, again there is no turning point and the universe recontracts back to zero size leading to a *Big Crunch*. Therefore, spatially closed universes ($k = +1$) with physical matter are also closed in time. All of these findings are summarised in Figure 22.

If the cosmological constant Λ is not zero, this correspondence "(open/closed) in space $\Leftrightarrow$ (open/closed) in time" is no longer necessarily true.

17.4 Density Parameters and the Critical Density

The primary purpose of this section is to introduce some convenient and commonly used notation and terminology in cosmology associated with the Friedmann equation (F1'). We will now include the cosmological constant in our analysis. For starters, however, let us again consider the case $\Lambda = 0$. (F1') can be written as

$$\frac{8\pi G\rho}{3H^2} - 1 = \frac{k}{a^2H^2} \ . \tag{17.12}$$

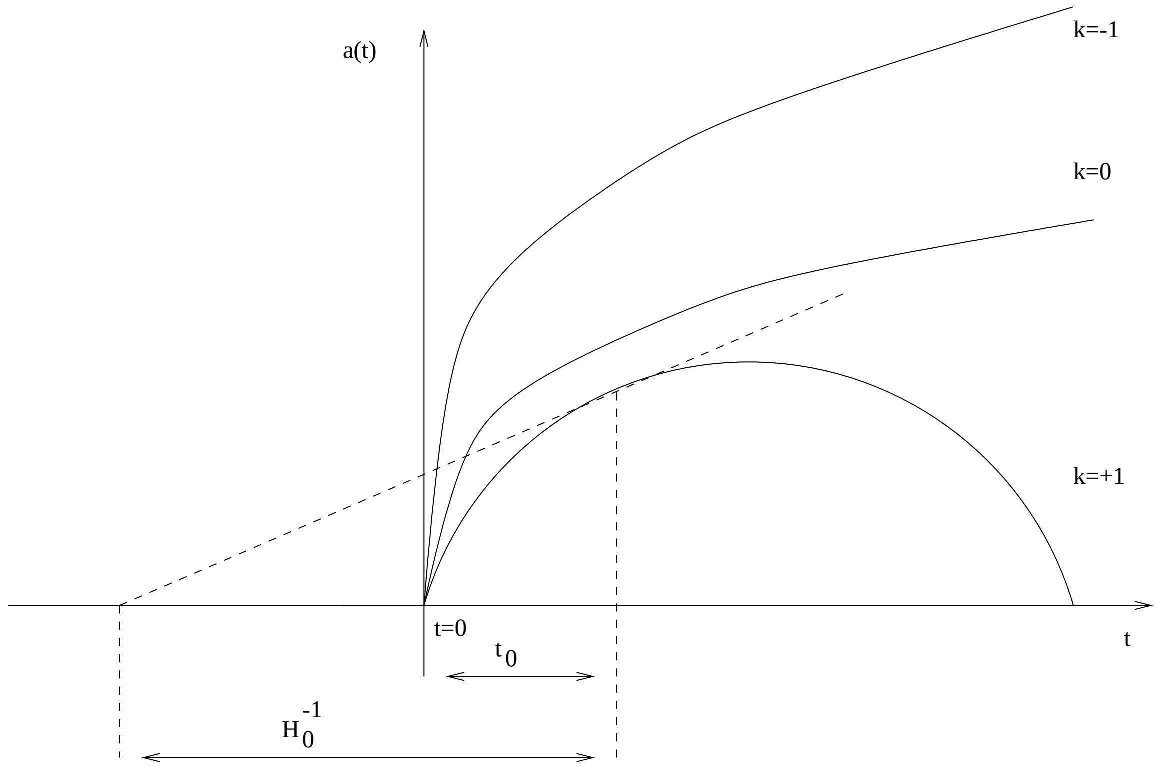

Figure 22: Qualitative behaviour of the Friedmann-Robertson-Walker models for $\Lambda = 0$. All models start with a Big Bang. For $k = +1$ the universe reaches a maximum radius and recollapses after a finite time. For $k = 0$, the universe keeps expanding but the expansion velocity tends to zero for $t \to \infty$ or $a \to \infty$. For $k = -1$, the expansion velocity approaches a non-zero constant value. Also shown is the significance of the Hubble time for the $k = +1$ universe showing clearly that H_0^{-1} gives an upper bound on the age of the universe.

If one defines the *critical density* ρ_{cr} by

$$\rho_{cr} = \frac{3H^2}{8\pi G} \quad , \tag{17.13}$$

and the *density parameter* Ω by

$$\Omega = \frac{\rho}{\rho_{cr}} \quad , \tag{17.14}$$

then (F1') becomes

$$\Omega - 1 = \frac{k}{a^2H^2} \tag{17.15}$$

Thus the sign of k is determined by whether the actual energy density ρ in the universe is greater than, equal to, or less than the critical density,

$$\begin{array}{ccccc} \rho < \rho_{cr} & \Leftrightarrow & k=-1 & \Leftrightarrow & \text{open} \\ \rho = \rho_{cr} & \Leftrightarrow & k=0 & \Leftrightarrow & \text{flat} \\ \rho > \rho_{cr} & \Leftrightarrow & k=+1 & \Leftrightarrow & \text{closed} \end{array}$$

This can be generalised to several species of (not mutually interacting) matter, characterised by equation of state parameters w_b, subject to the condition $w_b > 0$ or $w_b > -1/3$, with density parameters

$$\Omega_b = \frac{\rho_b}{\rho_{cr}} \quad . \tag{17.16}$$

The total matter contribution Ω_M is then

$$\Omega_M = \sum_b \Omega_b \quad . \tag{17.17}$$

Along the same lines we can also include the cosmological constant Λ. Indeed, inspection of the Friedmann equations reveals that the presence of a cosmological constant is equivalent to adding matter $(\rho_\Lambda, p_\Lambda)$ with

$$\Lambda \quad \Leftrightarrow \quad \rho_\Lambda = -p_\Lambda = \frac{\Lambda}{8\pi G} \quad w_\Lambda = -1 \quad . \tag{17.18}$$

Note that this identification is consistent with the conservation law (F3), since Λ is constant.

Then the Friedmann equation (F1') with a cosmological constant can be written as

$$(F1') \quad \Leftrightarrow \quad \Omega_M + \Omega_\Lambda = 1 + \frac{k}{a^2H^2} \quad , \tag{17.19}$$

where

$$\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{cr}} = \frac{\Lambda}{3H^2} \quad . \tag{17.20}$$

The 2nd order equation (F2') can also be written in terms of the density parameters,

$$q = \tfrac{1}{2}\sum_b (1+3w_b)\Omega_b - \Omega_\Lambda \quad . \tag{17.21}$$

Finally, one can also formally attribute an energy density ρ_k and pressure p_k to the curvature contribution $\sim k$ in the Friedmann equations. (F1), which does not depend on p, determines ρ_k, and then (F2), which does not depend on k, shows that $w_k = -1/3$ (so that $\rho_k + 3p_k = 0$). Thus the curvature contribution can be described as

$$k \quad \Leftrightarrow \quad \rho_k = -3p_k = \frac{-3k}{8\pi G a^2} \quad w_k = -1/3 \ , \tag{17.22}$$

with associated density parameter Ω_k. (F3) is identically satisfied in this case (or, if you prefer, requires that k is constant).

The Friedmann equation (F1) can now succinctly (if somewhat obscurely) be written as the condition that the sum of all density parameters be equal to 1,

$$(F1') \quad \Leftrightarrow \quad \Omega_M + \Omega_\Lambda + \Omega_k = 1 \ . \tag{17.23}$$

Clearly, it is thus of upmost importance to determine the various contributions ρ_a and ρ_Λ to the matter density ρ of the universe (and to determine ρ_{cr} e.g. by measurements of Hubble's constant).

17.5 The Universe Today

For a long time it was believed that the only non-negligible contribution to $\rho(t)$ today, let us call this $\rho_0 = \rho(t_0)$, is pressureless matter, $p_0 = 0$. If that were the case, then (F2') in the version (17.21) would imply

$$q_0 = \frac{\rho_0}{2\rho_{cr,0}} = \tfrac{1}{2}\Omega_M \ , \tag{17.24}$$

and thus k would be directly related to the value q_0 of the deceleration parameter today, as in

$$\begin{aligned} q_0 > 1/2 \quad &\Rightarrow \quad k = +1 \ , \quad \rho_0 > \rho_{cr} \\ q_0 < 1/2 \quad &\Rightarrow \quad k = -1 \ , \quad \rho_0 < \rho_{cr} \ . \end{aligned} \tag{17.25}$$

Moreover, observations indicated a value of ρ_0 much smaller than ρ_{cr}, thus suggesting a decelerating open $k = -1$ universe. While perhaps not the most hospitable place in the long run, at least this scenario had the virtue of simplicity.

However, exciting recent developments and observations in astronomy and astrophysics have provided strong evidence for a very different picture of the universe today. I will just the summarise the results here:

1. Current estimates for the matter contribution $\Omega_M = \sum_b \Omega_b$ are

$$\Omega_M \sim 0.3 \ . \tag{17.26}$$

2. Ordinary (visible, baryonic) matter only accounts for a small fraction of this, namely
$$\Omega_{M,\text{visible}} \sim 0.04 \quad . \tag{17.27}$$
Most of the matter density of the universe must therefore be due to some form of (as yet ill-understood) *dark matter*.

3. Independent observations from so-called *Supernovae* (as standard candles) and the study of the fine structure of the *Cosmic Microwave Background* both suggest that the universe is spatially flat, $k = 0$, and that the missing energy-density is due to (something that behaves very much like) a positive cosmological constant,
$$\Omega_\Lambda \sim 0.7 \qquad \Omega_M + \Omega_\Lambda = 1 \quad . \tag{17.28}$$

4. In particular, the cosmological constant is positive, leading to the conclusion that the universe is currently accelerating in its expansion rather than slowing down.

This currently favoured scenario raises all kinds of questions and puzzles, not just because of the dark matter component but, in particular, because of the presence of a cosmological constant whose energy density today is comparable to that of matter. An excellent recent introduction to cosmology which explains how the above results were obtained (and much more) is provided by the TASI lecture notes of M. Trodden and S. Carroll.[9]

[9] M. Trodden, S. Carroll, *TASI Lectures: Introduction to Cosmology*, `arXiv:astro-ph/0401547`.

18 Cosmology IV: Exact Solutions

18.1 Preliminaries

We have seen that a lot can be learnt about the Friedmann-Robertson-Walker models without ever having to solve a differential equation. On the other hand, more precise information can be obtained by specifying an equation of state for the matter content and solving the Friedmann equations.

In general, several species of matter, characterised by different equations of state or differnet equation of state parameters w_b will coexist. If we assume, as in the discussion of section 17.4, that these do not interact, then one can just add up their contributions in the Friedmann equations.

In order to make the dependence of the Friedmann equation (F1)

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - k + \frac{\Lambda}{3}a^2 \tag{18.1}$$

on the equation of state parameters w_b more manifest, it is useful to use the conservation law (16.40,17.6) to write

$$\frac{8\pi G}{3}\rho_b(t)a(t)^2 = C_b a(t)^{-(1+3w_b)} \tag{18.2}$$

for some constant C_b. Then the Friedmann equation takes the more explicit (in the sense that all the dependence on the cosmic scale factor $a(t)$ is explicit) form

$$\dot{a}^2 = \sum_b C_b a^{-(1+3w_b)} - k + \frac{\Lambda}{3}a^2 \quad . \tag{18.3}$$

In addition to the vacuum energy (and pressure) provided by Λ, there are typically two other kinds of matter which are relevant in our approximation, namely matter in the form of dust and radiation. Denoting the corresponding constants by C_m and C_r respectively, the Friedmann equation that we will be dealing with takes the form

$$(F1'') \quad \dot{a}^2 = \frac{C_m}{a} + \frac{C_r}{a^2} - k + \frac{\Lambda}{3}a^2 \quad , \tag{18.4}$$

illustrating the qualitatively different conntributions to the time-evolution. One can then characterise the different eras in the evolution of the universe by which of the above terms dominates, i.e. gives the leading contribution to the equation of motion for a. This already gives some insight into the physics of the situation.

We will call a universe

1. *matter dominated* if C_m/a dominates

2. *radiation dominated* if C_r/a^2 dominates

3. *curvature dominated* if k dominates
4. *vacuum dominated* if Λa^2 dominates

As mentioned before, for a long time it was believed that our present universe is purely matter dominated while recent observations appear to indicate that contributions from both matter and the cosmological constant are non-negligible.

Her are some immediate consequences of the Friedmann equation (F1"):

1. No matter how small C_r is, provided that it is non-zero, for sufficiently small values of a that term will dominate and one is in the radiation dominated era. In that case, one finds the characteristic behaviour

$$\dot{a}^2 = \frac{C_r}{a^2} \quad \Rightarrow \quad a(t) = (4C_r)^{1/4} t^{1/2} \quad . \tag{18.5}$$

It is more informative to trade the constant C_r for the condition $a(t_0) = a_0$, which leads to

$$a(t) = a_0 (t/t_0)^{1/2} \quad . \tag{18.6}$$

2. On the other hand, if C_m dominates, one has the characteristic behaviour

$$\dot{a}^2 = \frac{C_m}{a} \quad \Rightarrow \quad a(t) = (9C_m/4)^{1/3} t^{2/3} \tag{18.7}$$

or

$$a(t) = a_0 (t/t_0)^{2/3} \tag{18.8}$$

3. For general equation of state parameter $w \neq -1$, one similarly has

$$a(t) = a_0 (t/t_0)^h \; ; \qquad h = \frac{2}{3(1+w)} \quad . \tag{18.9}$$

This describes a decelerating universe ($h(h-1) < 0 \Rightarrow 0 < h < 1$) for $w > -1/3$ and an accelerating universe ($h > 1$) for $-1 < w < -1/3$.

4. For sufficiently large a, Λ, if not identically zero, will always dominate, no matter how small the cosmological constant may be, as all the other energy-content of the universe gets more and more diluted.

5. Only for $\Lambda = 0$ does k dominate for large a and one obtains, as we saw before, a constant expansion velocity (for $k = 0, -1$).

6. Finally, for $\Lambda = 0$ the Friedmann equation can be integrated in terms of elementary functions whereas for $\Lambda \neq 0$ one typically encounters elliptic integrals (unless $\rho = p = 0$).

18.2 The Einstein Universe

This particular solution is only of historical interest. Einstein was looking for a static cosmological solution and for this he was forced to introduce the cosmological constant.

Static means that $\dot{a} = 0$. Thus (F3) tells us that $\dot{\rho} = 0$. (F2) tells us that $4\pi G(\rho + 3p) = \Lambda$, where $\rho = \rho_m + \rho_r$. Therefore $p(t)$ also has to be time-independent, $\dot{p} = 0$, and moreover Λ has to be positive. We see that with $\Lambda = 0$ we would already not be able to satisfy this equation for physical matter content $\rho + 3p > 0$. From (F1") one deduces that

$$k = \frac{C_m}{a} + \frac{C_r}{a^2} + \frac{\Lambda}{3}a^2 \quad . \tag{18.10}$$

As all the terms on the right hand side are positive, this means that necessarily $k = +1$. Going back to (F1), setting $\dot{a} = 0$, $k = +1$, and substituting Λ by $4\pi G(\rho + 3p)$, one obtains a simple algebraic equation for $a(t) = a_0$, namely

$$\begin{aligned} a^2 &= (8\pi G\rho/3 + 4\pi G(\rho + 3p)/3)^{-1} \\ &= (4\pi G(\rho + p))^{-1} \quad . \end{aligned} \tag{18.11}$$

This is thus a static universe, with topology $\mathbb{R} \times S^3$ in which the gravitational attraction is precisely balanced by the cosmological constant. Note that even though a positive cosmological constant has a positive energy density, it has a negative pressure, and the net effect of a positive cosmological constant is that of gravitational repulsion rather than attraction.

18.3 The Matter Dominated Era

This is somewhat more realistic. In this case we have to solve

$$\dot{a}^2 = \frac{C_m}{a} - k \quad . \tag{18.12}$$

For $k = 0$, this is the equation we already discussed above, leading to the solution (18.8),

$$a(t) = a_0(t/t_0)^{2/3} \tag{18.13}$$

This solution is also known as the Einstein - de Sitter universe.

For $k = +1$, the equation is

$$\dot{a}^2 = \frac{C_m}{a} - 1 \quad . \tag{18.14}$$

We recall that in this case we will have a recollapsing universe with $a_{max} = C_m$, which is attained for $\dot{a} = 0$. This can be solved in closed form for t as a function of a, and the solution to

$$\frac{dt}{da} = (\frac{a}{a_{max} - a})^{-1/2} \tag{18.15}$$

is

$$t(a) = \frac{a_{max}}{2}\arccos(1-2a/a_{max}) - \sqrt{aa_{max}-a^2} \ , \tag{18.16}$$

as can easily be verified.

The universe starts at $t=0$ with $a(0)=0$, reaches its maximum $a=a_{max}$ at

$$t_{max} = a_{max}\arccos(-1)/2 = a_{max}\pi/2 \ , \tag{18.17}$$

and ends in a Big Crunch at $t = 2t_{max}$. The curve $a(t)$ is a cycloid, as is most readily seen by writing the solution in parametrised form. For this it is convenient to introduce the time-coordinate u via

$$\frac{du}{dt} = \frac{1}{a(t)} \ . \tag{18.18}$$

As an aside, note that with this time-coordinate the Robertson-Walker metric (for any k) takes the simple form

$$ds^2 = a^2(u)(-du^2 + d\tilde{s}^2) \ , \tag{18.19}$$

where again a tilde refers to the maximally symmetric spatial metric. In polar coordinates, this becomes

$$ds^2 = a^2(u)(-du^2 + d\psi^2 + f^2(\psi)d\Omega^2) \ . \tag{18.20}$$

Thus radial null lines are determined by $du = \pm d\psi$, as in flat space, and this coordinate system is very convenient for discussing the causal structure of the Friedmann-Robertson-Walker universes.

Anyway, in terms of the parameter u, the solution to the Friedmann equation for $k=+1$ can be written as

$$\begin{aligned} a(u) &= \frac{a_{max}}{2}(1-\cos u) \\ t(u) &= \frac{a_{max}}{2}(u-\sin u) \ , \end{aligned} \tag{18.21}$$

which makes it transparent that the curve is indeed a cycloid, roughly as indicated in Figure 22. The maximal radius is reached at

$$t_{max} = t(a=a_{max}) = t(u=\pi) = a_{max}\pi/2 \tag{18.22}$$

(with $a_{max} = C_m$), as before, and the total lifetime of the universe is $2t_{max}$.

We also see that for small times (for which matter dominates over curvature) the solution reduces to $t \sim u^3, a \sim u^2 \Rightarrow a \sim t^{2/3}$ which is the exact solution for $k=0$.

Analogously, for $k=-1$, the Friedmann equation can be solved in parametrised form, with the trignometric functions replaced by hyperbolic functions,

$$\begin{aligned} a(u) &= \frac{C_m}{2}(\cosh u - 1) \\ t(u) &= \frac{C_m}{2}(\sinh u - u) \ . \end{aligned} \tag{18.23}$$

18.4 The Radiation Dominated Era

In this case we need to solve

$$a^2\dot{a}^2 = C_r - ka^2 \quad . \tag{18.24}$$

Because a appears only quadratically, it is convenient to make the change of variables $b = a^2$. Then one obtains

$$\frac{\dot{b}^2}{4} + kb = C_r \quad . \tag{18.25}$$

For $k = 0$ we had already seen the solution in (18.6),

$$a(t) = a_0(t/t_0)^{1/2} \quad . \tag{18.26}$$

For $k = \pm 1$, one necessarily has $b(t) = b_0 + b_1 t + b_2 t^2$. Fixing $b(0) = 0$, one easily finds the solution

$$a(t) = [2C_r^{1/2}t - kt^2]^{1/2} \quad . \tag{18.27}$$

As expected this reduces to $a(t) \sim t^{1/2}$ for small times. For $k = +1$ one has

$$a(0) = a(2C_r^{1/2}) = 0 \quad . \tag{18.28}$$

Thus already electro-magnetic radiation is sufficient to shrink the universe again and make it recollapse. For $k = -1$, on the other hand, the universe expands forever. All this is of course in agreement with the results of the qualitative discussion given earlier.

18.5 The Vacuum Dominated Era: (Anti-) de Sitter Space

Even though not very realistic, this is of some interest for two reasons. On the one hand, as we know, Λ is the dominant driving force for a very large (and may therefore, if current observations are to be believed, dominate the late-time behaviour of our universe). On the other hand, recent cosmological models trying to solve the so-called *horizon problem* and *flatness problem* of the standard FRW model of cosmology use a mechanicsm called *inflation* and postulate a vacuum dominated era during some time in the early universe.

Thus the equation to solve is

$$\dot{a}^2 = -k + \frac{\Lambda}{3}a^2 \quad . \tag{18.29}$$

We see immediately that Λ has to be positive for $k = +1$ or $k = 0$, whereas for $k = -1$ both positive and negative Λ are possible.

This is one instance where the solution to the second order equation (F2),

$$\ddot{a} = \frac{\Lambda}{3}a \quad , \tag{18.30}$$

is more immediate, namely trigonometric functions for $\Lambda < 0$ (only possible for $k = -1$) and hyperbolic functions for $\Lambda > 0$. The first order equation then fixes the constants of integration according to the value of k.

For $k = 0$, the solution is obviously

$$a_{\pm}(t) = \sqrt{3/\Lambda}\mathrm{e}^{\pm\sqrt{\Lambda/3}t} \quad , \tag{18.31}$$

and for $k = +1$, thus $\Lambda > 0$, one has

$$a(t) = \sqrt{3/\Lambda}\cosh\sqrt{\Lambda/3}t \quad . \tag{18.32}$$

This is also known as the de Sitter universe. It is a maximally symmetric (in space-time) solution of the Einstein equations with a cosmological constant and thus has a metric of constant curvature (cf. the discussion in section 14). But we know that such a metric is unique. Hence the three solutions with $\Lambda > 0$, for $k = 0, \pm 1$ must all represent the same space-time metric, only in different coordinate systems (and it is a good exercise to check this explicitly). This is interesting because it shows that de Sitter space is so symmetric that it has spacelike slicings by three-spheres, by three-hyperboloids and by three-planes.

The solution for $k = -1$ involves $\sin\sqrt{|\Lambda|/3}t$ for $\Lambda < 0$ and $\sinh\sqrt{\Lambda/3}t$ for $\Lambda > 0$, as is easily checked. The former is known as the *anti de Sitter* universe.

19 Linearised Gravity and Gravitational Waves

19.1 Preliminary Remarks

In previous sections we have dealt with situations in General Relativity in which the gravitational field is strong and the full non-linearity of the Einstein equations comes into play (Black Holes, Cosmology). In most ordinary situations, however, the gravitational field is weak, very weak, and then it is legitimate to work with a linearisation of the Einstein equations. Our first aim will be to derive these linearised equations. As we will see, these turn out to be wave equations and we are thus naturally led to the subject of gravitational waves. These are an important prediction of General Relativity (there are no gravitational waves in Newton's theory). It is therefore important to understand how or under which circumstances they are created and how they can be detected. These, unfortunately, are rather complicated questions in general and I will not enter into this. The things I will cover in the following are much more elementary, both technically and conceptually, than anything else we have done recently.

19.2 The Linearised Einstein Equations

When we first derived the Einstein equations we checked that we were doing the right thing by deriving the Newtonian theory in the limit where

1. the gravitational field is weak
2. the gravitational field is static
3. test particles move slowly

We will now analyse a less restrictive situation in which we only impose the first condition. This is sufficient to deal with issues like gravitational waves and relativistic test-particles.

We express the weakness of the gravitational field by the condition that the metric be 'close' to that of Minkowski space, i.e. that

$$g_{\mu\nu} = g^{(1)}_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu} \tag{19.1}$$

with $|h_{\mu\nu}| \ll 1$. This means that we will drop terms which are quadratic or of higher power in $h_{\mu\nu}$. Here and in the following the superscript $^{(1)}$ indicates that we keep only up to linear (first order) terms in $h_{\mu\nu}$. In particular, the inverse metric is

$$g^{(1)\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{19.2}$$

where indices are raised with $\eta^{\mu\nu}$. As one has thus essentially chosen a background metric, the Minkowski metric, one can think of the linearised version of the Einstein equations (which are field equations for $h_{\mu\nu}$) as a Lorentz-invariant theory of a symmetric tensor field propagating in Minkowski space-time. I won't dwell on this but it is good to keep this in mind. It gives rise to the field theorist's picture of gravity as the theory of an interacting spin-2 field (which I do not subscribe to unconditionally because it is an inherently perturbative and background dependent picture).

It is straightforward to work out the Christoffel symbols and curvature tensors in this approximation. The terms quadratic in the Christoffel symbols do not contribute to the Riemann curvature tensor and one finds

$$\begin{aligned} \Gamma^{(1)\mu}_{\nu\lambda} &= \eta^{\mu\rho}\tfrac{1}{2}(\partial_\lambda h_{\rho\nu} + \partial_\nu h_{\rho\lambda} - \partial_\rho h_{\nu\lambda}) \\ R^{(1)}_{\mu\nu\rho\sigma} &= \tfrac{1}{2}(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\mu\partial_\sigma h_{\rho\nu} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\rho\mu}) \ . \end{aligned} \tag{19.3}$$

Hence the linearised Ricci tensor is

$$R^{(1)}_{\mu\nu} = \tfrac{1}{2}(\partial_\sigma\partial_\nu h^\sigma_{\ \mu} + \partial_\sigma\partial_\mu h^\sigma_{\ \nu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}) \ , \tag{19.4}$$

where $h = h^\mu_{\ \mu}$ is the trace of $h_{\mu\nu}$ and $\Box = \partial^\mu\partial_\mu$. Thus the Ricci scalar is

$$R^{(1)} = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h \ , \tag{19.5}$$

and the Einstein tensor is

$$G^{(1)}_{\mu\nu} = \tfrac{1}{2}(\partial_\sigma\partial_\nu h^\sigma_{\ \mu} + \partial_\sigma\partial_\mu h^\sigma_{\ \nu} - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu}\Box h) \ , \tag{19.6}$$

Therefore the linearised Einstein equations are

$$G^{(1)}_{\mu\nu} = 8\pi G T^{(0)}_{\mu\nu} \ . \tag{19.7}$$

Note that only the zero'th order term in the h-expansion appears on the right hand side of this equation. This is due to the fact that $T_{\mu\nu}$ must itself already be small in order for the linearised approximation to be valid, i.e. $T^{(0)}_{\mu\nu}$ should be of order $h_{\mu\nu}$. Therefore, any terms in $T_{\mu\nu}$ depending on $h_{\mu\nu}$ would already be of order $(h_{\mu\nu})^2$ and can be dropped.

Therefore the conservation law for the energy-momentum tensor is just

$$\partial_\mu T^{(0)\mu\nu} = 0 \ , \tag{19.8}$$

and this is indeed compatible with the linearised Bianchi identity

$$\partial_\mu G^{(1)\mu\nu} = 0 \ , \tag{19.9}$$

which can easily be verified. In fact, one has the stronger statement that

$$G^{(1)\mu\nu} = \partial_\rho Q^{\rho\mu\nu} \tag{19.10}$$

with $Q^{\rho\mu\nu} = -Q^{\mu\rho\nu}$, and this obviously implies the Bianchi identity.

19.3 Gauge Freedom and Coordinate Choices

To simplify life, it is now useful to employ the freedom we have in the choice of coordinates. What remains of general coordinate invariance in the linearised approximation are, naturally, linearised general coordinate transformations. Indeed, $h_{\mu\nu}$ and

$$h'_{\mu\nu} = h_{\mu\nu} + L_V \eta_{\mu\nu} \tag{19.11}$$

represent the same physical perturbation because $\eta_{\mu\nu} + L_V \eta_{\mu\nu}$ is just an infinitesimal coordinate transform of the Minkowski metric $\eta_{\mu\nu}$. Therefore linearised gravity has the gauge freedom

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu V_\nu + \partial_\nu V_\mu \quad . \tag{19.12}$$

For example, the linearised Riemann tensor $R^{(1)}_{\mu\nu\rho\sigma}$ is, rather obviously, invariant under this transformation (and hence so are the Einstein tensor etc.).

In general, a very useful gauge condition is

$$g^{\mu\nu}\Gamma^\rho_{\ \mu\nu} = 0 \quad . \tag{19.13}$$

It is called the *harmonic gauge condition* (or Fock, or de Donder gauge condition), and the name harmonic derives from the fact that in this gauge the coordinate fuctions x^μ are harmonic:

$$\Box x^\mu \equiv g^{\nu\rho}\nabla_\rho \partial_\nu x^\mu = -g^{\nu\rho}\Gamma^\mu_{\ \nu\rho} \quad , \tag{19.14}$$

and thus

$$\Box x^\mu = 0 \quad \Leftrightarrow \quad g^{\nu\rho}\Gamma^\mu_{\ \nu\rho} = 0 \quad . \tag{19.15}$$

It is the analogue of the Lorentz gauge $\partial_\mu A^\mu = 0$ in Maxwell theory. Moreover, in flat space Cartesian coordinates are obviously harmonic, and in general harmonic coordinates are (like geodesic coordinates) a nice and useful curved space counterpart of Cartesian coordinates.

In the linearised theory, this gauge condition becomes

$$\partial_\mu h^\mu_{\ \lambda} - \tfrac{1}{2}\partial_\lambda h = 0 \quad . \tag{19.16}$$

The gauge parameter V_μ which will achieve this is the solution to the equation

$$\Box V_\lambda = -(\partial_\mu h^\mu_{\ \lambda} - \tfrac{1}{2}\partial_\lambda h) \quad . \tag{19.17}$$

Indeed, with this choice one has

$$\partial_\mu (h^\mu_{\ \lambda} + \partial^\mu V_\lambda + \partial_\lambda V^\mu) - \tfrac{1}{2}(\partial_\lambda h + 2\partial^\mu V_\mu) = 0 \quad . \tag{19.18}$$

Note for later that, as in Maxwell theory, this gauge choice does not necessarily fix the gauge completely. Any transformation $x^\mu \to x^\mu + \xi^\mu$ with $\Box \xi^\mu = 0$ will leave the harmonic gauge condition invariant.

19.4 The Wave Equation

Now let us use this gauge condition in the linearised Einstein equations. In this gauge they simplify somewhat to

$$\Box h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\Box h = -16\pi G T^{(0)}_{\mu\nu} \ . \tag{19.19}$$

In particular, the vacuum equations are just

$$T^{(0)}_{\mu\nu} = 0 \ \Rightarrow \ \Box h_{\mu\nu} = 0 \ , \tag{19.20}$$

which is the standarad relativistic wave equation. Together, the equations

$$\begin{aligned} \Box h_{\mu\nu} &= 0 \\ \partial_\mu h^\mu_{\ \lambda} - \tfrac{1}{2}\partial_\lambda h &= 0 \end{aligned} \tag{19.21}$$

determine the evolution of a disturbance in a gravitational field in vacuum in the harmonic gauge.

It is often convenient to define the *trace reversed perturbation*

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h \ , \tag{19.22}$$

with

$$\bar{h}^\mu_{\ \mu} = -h^\mu_{\ \mu} \ . \tag{19.23}$$

Note, as an aside, that with this notation and terminology the Einstein tensor is the trace reversed Ricci tensor,

$$\bar{R}_{\mu\nu} = G_{\mu\nu} \ . \tag{19.24}$$

In terms of $\bar{h}_{\mu\nu}$, the Einstein equations and gauge conditions are just

$$\begin{aligned} \Box \bar{h}_{\mu\nu} &= -16\pi G T^{(0)}_{\mu\nu} \\ \partial_\mu \bar{h}^\mu_{\ \nu} &= 0 \ . \end{aligned} \tag{19.25}$$

In this equation, the $\bar{h}_{\mu\nu}$ are now decoupled. One solution is, of course, the retarded potential

$$\bar{h}_{\mu\nu}(\vec{x},t) = 4G\int d^3x' \, \frac{T^{(0)}_{\mu\nu}(\vec{x}', t - |\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|} \ . \tag{19.26}$$

Note that, as a consequence of $\partial_\mu T^{(0)\mu\nu} = 0$, this solution is automatically in the harmonic gauge.

The linearised vacuum Einstein equation in the harmonic gauge,

$$\Box\bar{h}_{\mu\nu} = 0 \ , \tag{19.27}$$

is clearly solved by

$$\bar{h}_{\mu\nu} = C_{\mu\nu}\mathrm{e}^{ik_\alpha x^\alpha} \ , \tag{19.28}$$

where $C_{\mu\nu}$ is a constant, symmetric *polarisation tensor* and k^α is a constant *wave vector*, provided that k^α is null, $k^\alpha k_\alpha = 0$. (In order to obtain real metrics one should of course use real solutions.)

Thus plane waves are solutions to the linearised equations of motion and the Einstein equations predict the existence of gravitational waves travelling along null geodesics (at the speed of light). The timelike component of the wave vector is often referred to as the *frequency* ω of the wave, and we can write $k^\mu = (\omega, k^i)$. Plane waves are of course not the most general solutions to the wave equations but any solution can be written as a superposition of plane wave solutions (wave packets).

So far, we have ten parameters $C_{\mu\nu}$ and four parameters k^μ to specify the wave, but many of these are spurious, i.e. can be eliminated by using the freedom to perform linearised coordinate transformations and Lorentz rotations.

First of all, the harmonic gauge condition implies that

$$\partial_\mu \bar{h}^\mu_{\ \nu} = 0 \quad \Rightarrow \quad k^\mu C_{\mu\nu} = 0 \ . \tag{19.29}$$

Now we can make use of the residual gauge freedom $x^\mu \to x^\mu + \xi^\mu$ with $\Box\xi^\mu = 0$ to impose further conditions on the polarisation tensor. Since this is a wave equation for ξ^μ, once we have specified a solution for ξ^μ we will have fixed the gauge completely. Taking this solution to be of the form

$$\xi^\mu = B_\mu \mathrm{e}^{ik_\alpha x^\alpha} \ , \tag{19.30}$$

one can choose the B_μ in such a way that the new polarisation tensor satisfies $k^\mu C_{\mu\nu} = 0$ (as before) as well as

$$C_{\mu 0} = C^\mu_{\ \mu} = 0 \ . \tag{19.31}$$

All in all, we appear to have nine conditions on the polarisation tensor $C_{\mu\nu}$ but as both (19.29) and the first of (19.31) imply $k^\mu C_{\mu 0} = 0$, only eight of these are independent. Therefore, there are two independent polarisations for a gravitational wave.

For example, we can choose the wave to travel in the x^3-direction. Then

$$k^\mu = (\omega, 0, 0, k^3) = (\omega, 0, 0, \omega) \ , \tag{19.32}$$

and $k^\mu C_{\mu\nu} = 0$ and $C_{0\nu} = 0$ imply $C_{3\nu} = 0$, so that the only independent components are C_{ab} with $a, b = 1, 2$. As C_{ab} is symmetric and traceless, this wave is completely characterised by $C_{11} = -C_{22}$, $C_{12} = C_{21}$ and the frequency ω.

Now we should not forget that, when talking about the polarisation tensor of a gravitational wave, we are actualy talking about the space-time metric itself. Namely, since for a traceless perturbation we have $\bar{h}_{\alpha\beta} = h_{\alpha\beta}$, we have deduced that the metric describing a gravitational wave travelling in the x^3-direction can always be put into the form

$$ds^2 = -dt^2 + (\delta_{ab} + h_{ab})dx^a dx^b + (dx^3)^2 \quad , \tag{19.33}$$

with $h_{ab} = h_{ab}(t \mp x^3)$.

19.6 Physical Effects of Gravitational Waves

To determine the physical effect of a gravitational wave racing by, we consider its influence on the relative motion of nearby particles. In other words, we look at the geodesic deviation equation. Consider a family of nearby particles described by the *velocity field* $u^\mu(x)$ and separation (deviation) vector $S^\mu(x)$,

$$\frac{D^2}{D\tau^2} S^\mu = R^\mu_{\ \nu\rho\sigma} u^\nu u^\rho S^\sigma \quad . \tag{19.34}$$

Now let us take the test particles to move slowly,

$$u^\mu = (1, 0, 0, 0) + \mathcal{O}(h) \quad . \tag{19.35}$$

Then, because the Riemann tensor is already of order h, the right hand side of the geodesic deviation equation reduces to

$$R^{(1)}_{\mu 00\sigma} = \tfrac{1}{2}\partial_0\partial_0 h_{\mu\sigma} \tag{19.36}$$

(because $h_{0\mu} = 0$). On the other hand, to lowest order the left hand side is just the ordinary time derivative. Thus the geodesic deviation equation becomes

$$\frac{\partial^2}{\partial t^2} S^\mu = \tfrac{1}{2} S^\sigma \frac{\partial^2}{\partial t^2} h^\mu_\sigma \quad . \tag{19.37}$$

In particular, we see immediately that the gravitational wave is *transversally polarised*, i.e. the component S^3 of S^μ in the longitudinal direction of the wave is unaffected and the particles are only disturbed in directions perpendicular to the wave. This gives rise to characteristic oscillating movements of the test particles in the 1-2 plane.

For example, with $C_{12} = 0$ one has

$$\begin{aligned} \frac{\partial^2}{\partial t^2} S^1 &= \tfrac{1}{2} S^1 \frac{\partial^2}{\partial t^2}(C_{11} e^{ik_\alpha x^\alpha}) \\ \frac{\partial^2}{\partial t^2} S^2 &= -\tfrac{1}{2} S^2 \frac{\partial^2}{\partial t^2}(C_{11} e^{ik_\alpha x^\alpha}) \quad . \end{aligned} \tag{19.38}$$

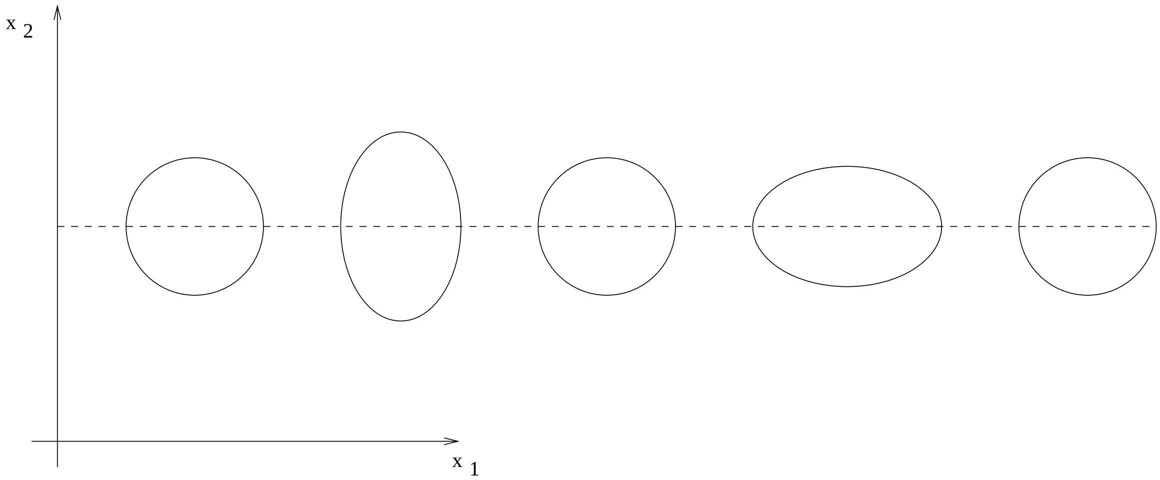

Figure 23: Effect of a gravitational wave with polarisation C_{11} moving in the x^3-direction, on a ring of test particles in the $x^1 - x^2$-plane.

Thus, to lowest order one has

$$\begin{aligned} S^1 &= (1 + \tfrac{1}{2} C_{11} \mathrm{e}^{ik_\alpha x^\alpha}) S^1(0) \\ S^2 &= (1 - \tfrac{1}{2} C_{11} \mathrm{e}^{ik_\alpha x^\alpha}) S^2(0) \end{aligned} \tag{19.39}$$

Recalling the interpretation of S^μ as a separation vector, this means that particles originally seperated in the x^1-direction will oscillate back and forth in the x^1-direction and likewise for x^2. A nice (and classical) way to visualise this (see Figure 23) is to start off with a ring of particles in the $1-2$ plane. As the wave passes by the particles will start bouncing in such a way that the ring bounces in the shape of a cross $+$. For this reason, C_{11} is also frequently written as C_+.

If, on the other hand, $C_{11} = 0$ but $C_{12} \neq 0$, then S^1 will be displaced in the direction of S^2 and vice versa,

$$\begin{aligned} S^1 &= S^1(0) + \tfrac{1}{2} C_{12} \mathrm{e}^{ik_\alpha x^\alpha} S^2(0) \\ S^2 &= S^2(0) + \tfrac{1}{2} C_{12} \mathrm{e}^{ik_\alpha x^\alpha} S^1(0) \ , \end{aligned} \tag{19.40}$$

and the ring of particles will bounce in the shape of a $\times$ ($C_{12} = C_\times$) - see Figure 24.

Of course, one can also construct circularly polarised waves by using

$$C_{R,L} = \frac{1}{\sqrt{2}} (C_{11} \pm i C_{12}) \ . \tag{19.41}$$

These solutions display the characteristic behaviour of *quadrupole radiation*, and this is something that we might have anticipated on general grounds. First of all, we know from Birkhoff's theorem that there can be no monopole (s-wave) radiation. Moreover,

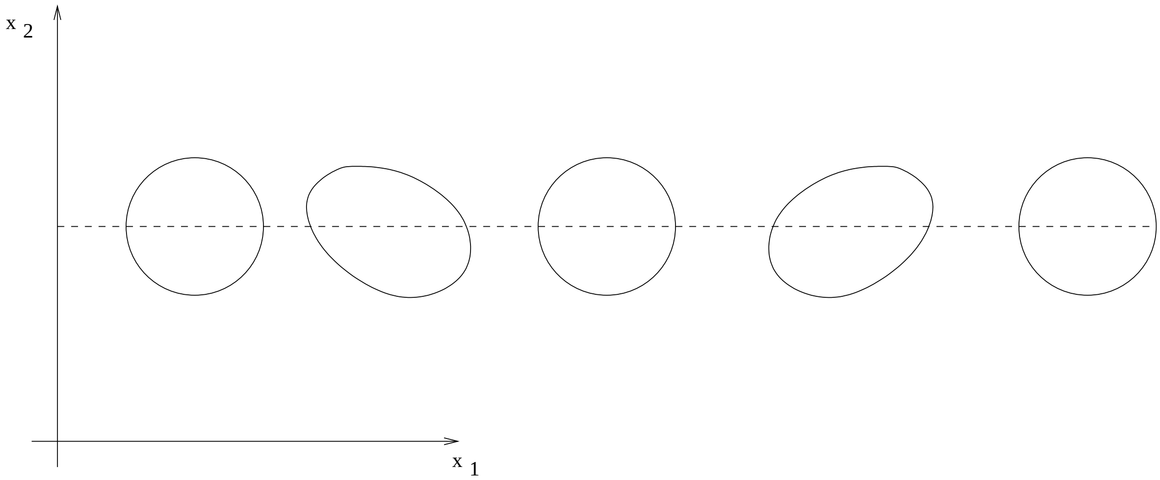

Figure 24: Effect of a gravitational wave with polarisation C_{12} moving in the x^3-direction, on a ring of test particles in the $x^1 - x^2$-plane.

dipole radiation is due to oscillations of the center of charge. While this is certainly possible for electric charges, an oscillation of the center of mass would violate momentum conservation and is therefore ruled out. Thus the lowest possible mode of gravitational radiation is quadrupole radiation, just as we have found.

19.7 Detection of Gravitational Waves

In principle, now that we have solutions to the vacuum equations, we should include sources and study the production of gravitational waves, characterise the type of radiation that is emitted, estimate the energy etc. I will not do this but just make some general comments on the detection of gravitational waves.

In principle, this ought to be straightforward. For example, one might like to simply try to track the separation of two freely suspended masses. Alternatively, the particles need not be free but could be connected by a solid piece of material. Then gravitational tidal forces will stress the material. If the resonant frequency of this 'antenna' equals the frequency of the gravitational wave, this should lead to a detectable oscillation. This is the principle of the so-called *Weber detectors* (1966-...), but these have not yet, as far as I know, produced completely conclusive results. In a sense this is not surprising as gravitational waves are extremely weak, so weak in fact that the quantum theory of the detectors (huge garbage can size aluminium cylinders, for example) needs to be taken into account.

However, there is indirect (and very compelling) evidence for gravitational waves. According to the theory (we have not developed), a binary system of stars rotating around its common center of mass should radiate gravitational waves (much like electromagnetic synchroton radiation). For two stars of equal mass M at distance $2r$ from each other, the prediction of General Relativity is that the power radiated by the binary system is

$$P = \frac{2}{5}\frac{G^4 M^5}{r^5} \quad . \tag{19.42}$$

This energy loss has actually been observed. In 1974, Hulse and Taylor discovered a binary system, affectionately known as PSR1913+16, in which both stars are very small and one of them is a *pulsar*, a rapidly spinning neutron star. The period of the orbit is only eight hours, and the fact that one of the stars is a pulsar provides a highly accurate clock with respect to which a change in the period as the binary loses energy can be measured. The observed value is in good agreement with the theoretical prediction for loss of energy by gravitational radiation and Hulse and Taylor were rewarded for these discoveries with the 1993 Nobel Prize.

Other situations in which gravitational waves might be either detected directly or inferred indirectly are extreme situations like gravitational collapse (supernovae) or matter orbiting black holes.

20 * Exact Wave-like Solutions of the Einstein Equations

In the previous section we discussed wave-like solutions to the linearised Einstein equations. In this section, we will briefly discuss a class of solutions to the full non-linear Einstein equations which are also wave-like and thus generalise the solutions of the previous section (to which they reduce in the weak-field limit). These solutions are called *plane-fronted waves with parallel rays* or *pp-waves* for short. A special subset of these solutions are the so-called *exact gravitational plane wave metrics* or simply *plane waves*.

Such wave-metrics have been studied in the context of four-dimensional general relativity for a long time even though they are not (and were never meant to be) phenomenologically realistic models of gravitational plane waves. The reason for this is that in the far-field gravitational waves are so weak that the linearised Einstein equations and their solutions are adequate to describe the physics, whereas the near-field strong gravitational effects responsible for the production of gravitational waves, for which the linearised equations are indeed insufficient, correspond to much more complicated solutions of the Einstein equations (describing e.g. two very massive stars orbiting around their common center of mass).

However, pp-waves have been useful and of interest as a theoretical play-ground since they are in some sense the simplest essentially Lorentzian metrics with no non-trivial Riemannian counterparts. As such they also provide a wealth of counterexamples to conjectures that one might like to make about Lorentzian geometry by naive extrapolation from the Riemannian case. They have also enjoyed some popularity in the string theory literature as potentially exact and exactly solvable string theory "backgrounds". However, they seem to have made it into very few textbook accounts of general relativity, and the purpose of this section is to at least partially fill this gap by providing a brief introduction to this topic.

20.1 Plane Waves in Rosen Coordinates: Heuristics

We have seen in the previous section that a metric describing the propagation of a plane wave in the x^3-direction (19.33) can be written as

$$ds^2 = -dt^2 + (\delta_{ab} + h_{ab})dx^a dx^b + (dx^3)^2 \quad , \tag{20.1}$$

with $h_{ab} = h_{ab}(t - x^3)$.

In terms of lightcone coordinates $U = z - t$, $V = (z + t)/2$ this can be written as

$$ds^2 = 2dUdV + (\delta_{ij} + h_{ij}(U))dy^i dy^j \quad . \tag{20.2}$$

We will now simply define a plane wave metric in general relativity to be a metric of the above form, dropping the assumption that h_{ij} be "small",

$$d\bar{s}^2 = 2dUdV + \bar{g}_{ij}(U)dy^i dy^j \quad . \tag{20.3}$$

We will say that this is a plane wave metric in *Rosen coordinates*. This is not the coordinate system in which plane waves are usually discussed, among other reasons because typically in Rosen coordinates the metric exhibits spurious coordinate singularities. This led to the mistaken belief in the past that there are no non-singular plane wave solutions of the non-linear Einstein equations. We will establish the relation to the more common and much more useful *Brinkmann coordinates* below.

Plane wave metrics are characterised by a single matrix-valued function of U, but two metrics with quite different $\bar{g}_{ij}$ may well be isometric. For example,

$$d\bar{s}^2 = 2dUdV + U^2 d\vec{y}^2 \tag{20.4}$$

is isometric to the flat Minkowski metric whose natural presentation in Rosen coordinates is simply the Minkowski metric in lightcone coordinates,

$$d\bar{s}^2 = 2dUdV + d\vec{y}^2 \quad . \tag{20.5}$$

This is not too difficult to see, and we will establish this as a consequence of a more general result in section 20.8 (but if you want to try this now, try scaling $\vec{y}$ by U and do something to V ...).

That (20.4) is indeed flat should in any case not be too surprising. It is the "null" counterpart of the "spacelike" fact that $ds^2 = dr^2 + r^2 d\Omega^2$, with $d\Omega^2$ the unit line element on the sphere, is just the flat Euclidean metric in polar coordinates, and the "timelike" statement that

$$ds^2 = -dt^2 + t^2 d\tilde{\Omega}^2 \quad , \tag{20.6}$$

with $d\tilde{\Omega}^2$ the unit line element on the hyperboloid, is just (a wedge of) the flat Minkowski metric. In cosmology this is known as the *Milne Universe*. It is easy to check that this is indeed a (rather trivial) solution of the Friedmann equations with $k = -1$, $a(t) = t$ and $\rho = P = 0$.

It is somewhat less obvious, but still true, that for example the two metrics

$$\begin{aligned} d\bar{s}^2 &= 2dUdV + \sinh^2 U d\vec{y}^2 \\ d\bar{s}^2 &= 2dUdV + \mathrm{e}^{2U} d\vec{y}^2 \end{aligned} \tag{20.7}$$

are also isometric.

20.2 From pp-waves to plane waves in Brinkmann coordinates

In the remainder of this section we will study gravitational plane waves in a more systematic way. One of the characteristic features of the above plane wave metrics is the existence of a nowhere vanishing covariantly constant null vector field, namely ∂_V. We thus begin by deriving the general metric (line element) for a space-time admitting such a covariantly constant null vector field. We will from now on consider general $(d+2)$-dimensional space-times, where d is the number of transverse dimensions.

Thus, let Z be a parallel (i.e. covariantly constant) null vector of the $(d+2)$-dimensional Lorentzian metric $g_{\mu\nu}$, $\nabla_\mu Z^\nu = 0$. This condition is equivalent to the pair of conditions

$$\nabla_\mu Z_\nu + \nabla_\nu Z_\mu = 0 \tag{20.8}$$

$$\nabla_\mu Z_\nu - \nabla_\nu Z_\mu = 0 \ . \tag{20.9}$$

The first of these says that Z is a Killing vector field, and the second that Z is also a gradient vector field. If Z is nowhere zero, without loss of generality we can assume that

$$Z = \partial_v \tag{20.10}$$

for some coordinate v since this simply means that we are using a parameter along the integral curves of Z as our coordinate v. In terms of components this means that $Z^\mu = \delta^\mu_v$, or

$$Z_\mu = g_{\mu v} \ . \tag{20.11}$$

The fact that Z is null means that

$$Z_v = g_{vv} = 0 \ . \tag{20.12}$$

The Killing equation now implies that all the components of the metric are v-independent,

$$\partial_v g_{\mu\nu} = 0 \ . \tag{20.13}$$

The second condition (20.9) is identical to

$$\nabla_\mu Z_\nu - \nabla_\nu Z_\mu = 0 \ \Leftrightarrow \ \partial_\mu Z_\nu - \partial_\nu Z_\mu = 0 \ , \tag{20.14}$$

which implies that locally we can find a function $u = u(x^\mu)$ such that

$$Z_\mu = g_{v\mu} = \partial_\mu u \ . \tag{20.15}$$

There are no further constraints, and thus the general form of a metric admitting a parallel null vector is, changing from the x^μ-coordinates to $\{u, v, x^a\}$, $a = 1, \ldots, d$,

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= 2dudv + g_{uu}(u,x^c)du^2 + 2g_{au}(u,x^c)dx^a du + g_{ab}(u,x^c)dx^a dx^b \\ &\equiv 2dudv + K(u,x^c)du^2 + 2A_a(u,x^c)dx^a du + g_{ab}(u,x^c)dx^a dx^b \ . \end{aligned} \tag{20.16}$$

Note that if we had considered a metric with a covariantly constant timelike or spacelike vector, then we would have obtained the above metric with an additional term of the form $\mp dv^2$. In that case, the cross-term $2dudv$ could have been eliminated by shifting $v \to v' = v \mp u$, and the metric would have factorised into $\mp dv'^2$ plus a v'-independent metric. Such a factorisation does in general not occur for a covariantly constant null vector, which makes metrics with such a vector potentially more interesting than their timelike or spacelike counterparts.

There are still residual coordinate transformations which leave the above form of the metric invariant. For example, both K and A_a can be eliminated in favour of g_{ab}. We will not pursue this here, as we are primarily interested in a special class of metrics which are characterised by the fact that $g_{ab} = \delta_{ab}$,

$$ds^2 = 2dudv + K(u,x^b)du^2 + 2A_a(u,x^b)dx^a du + d\vec{x}^2 \quad . \tag{20.17}$$

Such metrics are called *plane-fronted waves with parallel rays*, or *pp-waves* for short. "plane-fronted" refers to the fact that the wave fronts $u = const.$ are planar (flat), and "parallel rays" refers to the existence of a parallel null vector. Once again, there are residual coordinate transformations which leave this form of the metric invariant. Among them are shifts of v, $v \to v + \Lambda(u,x^a)$, under which the coefficients K and A_a transform as

$$\begin{aligned} K &\to K + \tfrac{1}{2}\partial_u\Lambda \\ A_a &\to A_a + \partial_a\Lambda \quad . \end{aligned} \tag{20.18}$$

Note in particular the "gauge transformation" of the (Kaluza-Klein) gauge field A_a, here associated with the null isometry generated by $Z = \partial_v$.

Plane waves are a very special kind of pp-waves. By definition, a plane wave metric is a pp-wave with $A_a = 0$ and $K(u,x^a)$ quadratic in the x^a (zero'th and first order terms in x^a can be eliminated by a coordinate transformation),

$$d\bar{s}^2 = 2dudv + A_{ab}(u)x^a x^b du^2 + d\vec{x}^2 \quad . \tag{20.19}$$

We will say that this is the metric of a plane wave in *Brinkmann coordinates*. The relation between the expressions for a plane wave in Brinkmann coordinates and Rosen coordinates will be explained in section 20.8. From now on barred quantities will refer to plane wave metrics.

In Brinkmann coordinates a plane wave metric is characterised by a single symmetric matrix-valued function $A_{ab}(u)$. Generically there is very little redundancy in the description of plane waves in Brinkmann coordinates, i.e. there are very few residual coordinate transformations that leave the form of the metric invariant, and the metric is specified almost uniquely by $A_{ab}(u)$. In particular, as we will see below, a plane wave metric is flat if and only if $A_{ab}(u) = 0$ identically. Contrast this with the non-uniqueness of the flat metric in Rosen coordinates. This uniqueness of the Brinkmann coordinates is one of the features that makes them convenient to work with in concrete applications.

20.3 Geodesics, Light-Cone Gauge and Harmonic Oscillators

We now take a look at geodesics of a plane wave metric in Brinkmann coordinates,

$$d\bar{s}^2 = 2dudv + A_{ab}(u)x^a x^b du^2 + d\vec{x}^2 \ , \tag{20.20}$$

i.e. the solutions $x^\mu(\tau)$ to the geodesic equations

$$\ddot{x}^\mu(\tau) + \bar{\Gamma}^\mu_{\nu\lambda}(x(\tau))\dot{x}^\nu(\tau)\dot{x}^\lambda(\tau) = 0 \ , \tag{20.21}$$

where an overdot denotes a derivative with respect to the affine parameter τ.

Rather than determining the geodesic equations by first calculating all the non-zero Christoffel symbols, we make use of the fact that the geodesic equations can be obtained more efficiently, and in a way that allows us to directly make use of the symmetries of the problem, as the Euler-Lagrange equations of the Lagrangian

$$\begin{aligned} \mathcal{L} &= \tfrac{1}{2}\bar{g}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \\ &= \dot{u}\dot{v} + \tfrac{1}{2}A_{ab}(u)x^a x^b \dot{u}^2 + \tfrac{1}{2}\dot{\vec{x}}^2 \ , \end{aligned} \tag{20.22}$$

supplemented by the constraint

$$2\mathcal{L} = \epsilon \ , \tag{20.23}$$

where $\epsilon = 0$ ($\epsilon = -1$) for massless (massive) particles.

Since nothing depends on v, the *lightcone momentum*

$$p_v = \frac{\partial \mathcal{L}}{\partial \dot{v}} = \dot{u} \tag{20.24}$$

is conserved. For $p_v = 0$ the particle obviously does not feel the curvature and the geodesics are straight lines. When $p_v \neq 0$, we choose the *lightcone gauge*

$$u = p_v \tau \ . \tag{20.25}$$

Then the geodesic equations for the transverse coordinates are the Euler-Lagrange equations

$$\ddot{x}^a(\tau) = A_{ab}(p_v\tau)x^b(\tau)p_v^2 \tag{20.26}$$

These are the equation of motion of a non-relativistic *harmonic oscillator*,

$$\ddot{x}^a(\tau) = -\omega^2_{ab}(\tau)x^b(\tau) \tag{20.27}$$

with (possibly time-dependent) frequency matrix

$$\omega^2_{ab}(\tau) = -p_v^2 A_{ab}(p_v\tau) \ , \tag{20.28}$$

The constraint

$$p_v\dot{v}(\tau) + \tfrac{1}{2}A_{ab}(p_v\tau)x^a(\tau)x^b(\tau)p_v^2 + \tfrac{1}{2}\dot{x}^a(\tau)\dot{x}^a(\tau) = 0 \tag{20.29}$$

for null geodesics (the case $\epsilon \neq 0$ can be dealt with in the same way) implies, and thus provides a first integral for, the v-equation of motion. Multiplying the oscillator equation by x^a and inserting this into the constraint, one finds that this can be further integrated to

$$p_v v(\tau) = -\tfrac{1}{2} x^a(\tau)\dot{x}^a(\tau) + p_v v_0 \quad . \tag{20.30}$$

Note that a particular solution of the null geodesic equation is the purely "longitudinal" null geodesic

$$x^\mu(\tau) = (u = p_v \tau, v = v_0, x^a = 0) \quad . \tag{20.31}$$

Along this null geodesic, all the Christoffel symbols of the metric (in Brinkmann coordinates) are zero. Hence Brinkmann coordinates can be regarded as a special case of *Fermi coordinates* (briefly mentioned at the beginning of section 2.7).

By definition the lightcone Hamiltonian is

$$H_{lc} = -p_u \quad , \tag{20.32}$$

where p_u is the momentum conjugate to u in the gauge $u = p_v \tau$. With the above normalisation of the Lagrangian one has

$$\begin{aligned} p_u &= \bar{g}_{u\mu}\dot{x}^\mu = \dot{v} + A_{ab}(p_v\tau)x^a x^b p_v \\ &= -p_v^{-1} H_{ho}(\tau) \quad , \end{aligned} \tag{20.33}$$

where $H_{ho}(\tau)$ is the (possibly time-dependent) harmonic oscillator Hamiltonian

$$H_{ho}(\tau) = \tfrac{1}{2}(\dot{x}^a\dot{x}^a - p_v^2 A_{ab}(p_v\tau)x^a x^b) \quad . \tag{20.34}$$

Thus for the lightcone Hamiltonian one has

$$H_{lc} = \tfrac{1}{p_v} H_{ho} \quad . \tag{20.35}$$

In summary, we note that in the lightcone gauge the equation of motion for a relativistic particle becomes that of a non-relativistic harmonic oscillator. This harmonic oscillator equation appears in various different contexts when discussing plane waves, and will therefore also reappear several times later on in this section.

20.4 Curvature and Singularities of Plane Waves

It is easy to see that there is essentially only one non-vanishing component of the Riemann curvature tensor of a plane wave metric, namely

$$\bar{R}_{uaub} = -A_{ab} \quad . \tag{20.36}$$

In particular, therefore, because of the null (or chiral) structure of the metric, there is only one non-trivial component of the Ricci tensor,

$$\bar{R}_{uu} = -\delta^{ab}A_{ab} \equiv -\operatorname{Tr} A \quad , \tag{20.37}$$

the Ricci scalar is zero,

$$\bar{R} = 0 \ , \tag{20.38}$$

and the only non-zero component of the Einstein tensor (7.53) is

$$\bar{G}_{uu} = \bar{R}_{uu} \ . \tag{20.39}$$

Thus, as claimed above, the metric is flat iff $A_{ab} = 0$. Moreover, we see that in Brinkmann coordinates the vacuum Einstein equations reduce to a simple algebraic condition on A_{ab} (regardless of its u-dependence), namely that it be traceless.

A simple example of a vacuum plane wave metric in four dimensions is

$$d\bar{s}^2 = 2dudv + (x^2 - y^2)du^2 + dx^2 + dy^2 \ , \tag{20.40}$$

or, more generally,

$$d\bar{s}^2 = 2dudv + [A(u)(x^2 - y^2) + 2B(u)xy]du^2 + dx^2 + dy^2 \tag{20.41}$$

for arbitrary fuctions $A(u)$ and $B(u)$. This reflects the two polarisation states or degrees of freedom of a four-dimensional graviton. Evidently, this generalises to arbitrary dimensions: the number of degrees of freedom of the traceless matrix $A_{ab}(u)$ correspond precisely to those of a transverse traceless symmetric tensor (a.k.a. a graviton).

The Weyl tensor is the traceless part of the Riemann tensor,

$$\bar{C}_{uaub} = -(A_{ab} - \frac{1}{d}\delta_{ab} \operatorname{Tr} A) \ . \tag{20.42}$$

Thus the Weyl tensor vanishes (and, for $d > 1$, the plane wave metric is conformally flat) iff A_{ab} is pure trace,

$$A_{ab}(u) = A(u)\delta_{ab} \ . \tag{20.43}$$

For $d = 1$, every plane wave is conformally flat, as is most readily seen in Rosen coordinates.

When the Ricci tensor is non-zero (A_{ab} has non-vanishing trace), then plane waves solve the Einstein equations with null matter or null fluxes, i.e. with an energy-momentum tensor $\bar{T}_{\mu\nu}$ whose only non-vanishing component is $\bar{T}_{uu}$,

$$\bar{T}_{\mu\nu} = \rho(u)\delta_{\mu u}\delta_{\nu u} \ . \tag{20.44}$$

Examples are e.g. null Maxwell fields $A_\mu(u)$ with field strength

$$F_{u\mu} = -F_{\mu u} = \partial_u A_\mu \ . \tag{20.45}$$

Physical matter (with positive energy density) corresponds to $\bar{R}_{uu} > 0$ or $\operatorname{Tr} A < 0$.

It is pretty obvious by inspection that not just the scalar curvature but all the scalar curvature invariants of a plane wave, i.e. scalars built from the curvature tensor and its covariant derivatives, vanish since there is simply no way to soak up the u-indices.

Usually, an unambiguous way to ascertain that what appears to be a singularity of a metric is a true curvature singularity rather than just a singularity in the choice of coordinates is to exhibit a curvature invariant that is singular at that point. For example, for the Schwarzschild metric one has the Kretschmann scalar (13.83) $K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \sim m^2/r^6$, which shows that the singularity at $r = 0$ is a true singularity.

Now for plane waves all curvature invariants are zero. Does this mean that plane waves are non-singular? Or, if not, how does one detect the presence of a curvature singularity? One way to do this is to study the tidal forces acting on extended objects or families of freely falling particles. Indeed, in a certain sense the main effect of curvature (or gravity) is that initially parallel trajectories of freely falling non-interacting particles (dust, pebbles,...) do not remain parallel, i.e. that gravity has the tendency to focus (or defocus) matter. This statement find its mathematically precise formulation in the geodesic deviation equation (8.28),

$$\frac{D^2}{D\tau^2}\delta x^\mu = R^\mu_{\ \nu\lambda\rho}\dot{x}^\nu\dot{x}^\lambda\delta x^\rho \quad . \tag{20.46}$$

Here δx^μ is the separation vector between nearby geodesics. We can apply this equation to some family of geodesics of plane waves discussed in section 20.3. We will choose δx^μ to connect points on nearby geodesics with the same value of $\tau = u$. Thus $\delta u = 0$, and the geodesic deviation equation for the transverse separations δx^a reduces to

$$\frac{d^2}{du^2}\delta x^a = -\bar{R}^a_{\ ubu}\delta x^b = A_{ab}\delta x^b \quad . \tag{20.47}$$

This is (once again!) the harmonic oscillator equation. We could have also obtained this directly by varying the harmonic oscillator (geodesic) equation for x^a, using $\delta u = 0$. We see that for negative eigenvalues of A_{ab} (physical matter) this tidal force is attractive, leading to a focussing of the geodesics. For vacuum plane waves, on the other hand, the tidal force is attractive in some directions and repulsive in the other (reflecting the quadrupole nature of gravitational waves).

What is of interest to us here is the fact that the above equation shows that A_{ab} itself contains direct physical information. In particular, these tidal forces become infinite where $A_{ab}(u)$ diverges. This is a true physical effect and hence the plane wave space-time is genuinely singular at such points.

Let us assume that such a singularity occurs at $u = u_0$. Since $u = p_v\tau$ is an affine parameter along the geodesic, this shows that any geodesic starting off at a finite value u_1 of u will reach the singularity in the finite "time" $u_0 - u_1$. Thus the space-time is geodesically incomplete and ends at $u = u_0$.

Since, on the other hand, the plane wave metric is clearly smooth for non-singular $A_{ab}(u)$, we can thus summarise this discussion by the statement that a plane wave is singular if and only if $A_{ab}(u)$ is singular somewhere.

I still owe you an explanation of what the heuristic considerations of section 20.1 have to do with the rest of this section. To that end I will now describe the relation between the plane wave metric in Brinkmann coordinates,

$$d\bar{s}^2 = 2dudv + A_{ab}(u)x^a x^b du^2 + d\vec{x}^2 \quad , \tag{20.48}$$

and in Rosen coordinates,

$$d\bar{s}^2 = 2dUdV + \bar{g}_{ij}(U)dy^i dy^j \quad . \tag{20.49}$$

It is clear that, in order to transform the non-flat transverse metric in Rosen coordinates to the flat transverse metric in Brinkmann coordinates, one should change variables as

$$x^a = \bar{E}^a_i y^i \quad , \tag{20.50}$$

where $\bar{E}^a_i$ is a "vielbein" for $\bar{g}_{ij}$, i.e. it is a matrix which satisfies

$$\bar{g}_{ij} = \bar{E}^a_i \bar{E}^b_j \delta_{ab} \quad . \tag{20.51}$$

Denoting the inverse vielbein by $\bar{E}^i{}_a$, one has

$$\bar{g}_{ij}dy^i dy^j = (dx^a - \dot{\bar{E}}^a_i \bar{E}^i{}_c x^c dU)(dx^b - \dot{\bar{E}}^b{}_j \bar{E}^j{}_d x^d dU)\delta_{ab} \quad . \tag{20.52}$$

This generates the flat transverse metric as well as dU^2-term quadratic in the x^a, as desired, but there are also unwanted $dUdx^a$ cross-terms. Provided that $\bar{E}$ satisfies the symmetry condition

$$\dot{\bar{E}}_{ai}\bar{E}^i{}_b = \dot{\bar{E}}_{bi}\bar{E}^i{}_a \tag{20.53}$$

(such an $\bar{E}$ can always be found and is unique up to U-independent orthogonal transformations), these terms can be cancelled by a shift in V,

$$V \to V - \tfrac{1}{2}\dot{\bar{E}}_{ai}\bar{E}^i{}_b x^a x^b \quad . \tag{20.54}$$

Apart from eliminating the $dUdx^a$-terms, this shift will also have the effect of generating other dU^2-terms. Thanks to the symmetry condition, the term quadratic in first derivatives of $\bar{E}$ cancels that arising from $\bar{g}_{ij}dy^i dy^j$, and only a second-derivative part remains. The upshot of this is that after the change of variables

$$\begin{aligned} U &= u \\ V &= v + \tfrac{1}{2}\dot{\bar{E}}_{ai}\bar{E}^i{}_b x^a x^b \\ y^i &= \bar{E}^i{}_a x^a \quad , \end{aligned} \tag{20.55}$$

the metric (20.49) takes the Brinkmann form (20.48), with

$$A_{ab} = \ddot{\bar{E}}_{ai}\bar{E}^i{}_b \quad . \tag{20.56}$$

This can also be written as the *harmonic oscillator equation*

$$\ddot{\bar{E}}_{ai} = A_{ab}\bar{E}_{bi} \tag{20.57}$$

we had already encountered in the context of the geodesic (and geodesic deviation) equation.

Note that from this point of view the Rosen coordinates are labelled by d out of $2d$ linearly independent solutions of the oscillator equation, and the symmetry condition can now be read as the constraint that the *Wronskian* among these solutions be zero. Thus, given the metric in Brinkmann coordinates, one can construct the metric in Rosen coordinates by solving the oscillator equation, choosing a maximally commuting set of solutions to construct $\bar{E}_{ai}$, and then determining $\bar{g}_{ij}$ algebraically from the $\bar{E}_{ai}$.

In practice, once one knows that Rosen and Brinkmann coordinates are indeed just two distinct ways of describing the same class of metrics, one does not need to perform explicitly the coordinate transformation mapping one to the other. All one is interested in is the above relation between $\bar{g}_{ij}(U)$ and $A_{ab}(u)$, which essentially says that A_{ab} is the curvature of $\bar{g}_{ij}$,

$$A_{ab} = -\bar{E}^i{}_a\bar{E}^j{}_b\bar{R}_{UiUj} \quad . \tag{20.58}$$

The equations simplify somewhat when the metric $\bar{g}_{ij}(u)$ is diagonal,

$$\bar{g}_{ij}(u) = \bar{e}_i(u)^2\delta_{ij} \quad . \tag{20.59}$$

In that case one can choose $\bar{E}^a_i = \bar{e}_i\delta^a_i$. The symmetry condition is automatically satisfied because a diagonal matrix is symmetric, and one finds that A_{ab} is also diagonal,

$$A_{ab} = (\ddot{\bar{e}}_a/\bar{e}_a)\delta_{ab} \quad . \tag{20.60}$$

Conversely, therefore, given a diaognal plane wave in Brinkmann coordinates, to obtain the metric in Rosen coordinates one needs to solve the harmonic oscillator equations

$$\ddot{\bar{e}}_i(u) = A_{ii}(u)\bar{e}_i(u) \quad . \tag{20.61}$$

Thus the Rosen metric determined by $\bar{g}_{ij}(U)$ is flat iff $\bar{e}_i(u) = a_iU + b_i$ for some constants a_i, b_i. In particular, we recover the fact that the metric (20.4),

$$d\bar{s}^2 = 2dUdV + U^2d\vec{y}^2 \tag{20.62}$$

is flat. We see that the non-uniqueness of the metric in Rosen coordinates is due to the integration 'constants' arising when trying to integrate a curvature tensor to a corresponding metric.

As another example, consider the four-dimensional vacuum plane wave (20.40). Evidently, one way of writing this metric in Rosen coordinates is

$$d\bar{s}^2 = 2dUdV + \sinh^2 U dX^2 + \sin^2 U dY^2 \quad , \tag{20.63}$$

and more generally any plane wave with constant A_{ab} can be chosen to be of this trigonometric form in Rosen coordinates.

20.6 More on Rosen Coordinates

Collecting the results of the previous sections, we can now gain a better understanding of the geometric significance (and shortcomings) of Rosen coordinates for plane waves.

First of all we observe that the metric

$$d\bar{s}^2 = 2dUdV + \bar{g}_{ij}(U)dy^i dy^j \tag{20.64}$$

defines a preferred family (congruence) of null geodesics, namely the integral curves of the null vector field ∂_U, i.e. the curves

$$(U(\tau), V(\tau), y^k(\tau)) = (\tau, V, y^k) \tag{20.65}$$

with affine parameter $\tau = U$ and parametrised by the constant values of the coordinates (V, y^k). In particular, the "origin" $V = y^k = 0$ of this congruence is the longitudinal null geodesic (20.31) with $v_0 = 0$ in Brinkmann coordinates.

In the region of validity of this coordinate system, there is a unique null geodesic of this congruence passing through any point, and one can therefore label (coordinatise) these points by specifying the geodesic (V, y^k) and the affine parameter U along that geodesic, i.e. by Rosen coordinates.

We can now also understand the reasons for the failure of Rosen coordinates: they cease to be well-defined (and give rise to spurious coordinate singularities) e.g. when geodesics in the family (congruence) of null geodesics interesect: in that case there is no longer a unique value of the coordinates (U, V, y^k) that one can associate to that intersection point.

To illustrate this point, consider simply $\mathbb{R}^2$ with its standard metric $ds^2 = dx^2 + dy^2$. An example of a "good" congruence of geodesics is the straight lines parallel to the x-axis. The corresponding "Rosen" coordinates ("Rosen" in quotes because we are not talking about null geodesics) are simply the globally well-defined Cartesian coordinates, x playing the role of the affine parameter U and y that of the transverse coordinates y^k labelling the geodesics. An example of a "bad" family of godesics is the straight lines through the origin. The corresponding "Rosen" coordinates are essentially just polar coordinates. Away from the origin there is again a unique geodesic passing through any point but, as is well known, this coordinate system breaks down at the origin.

With this in mind, we can now reconsider the "bad" Rosen coordinates

$$d\bar{s}^2 = 2dUdV + U^2 d\vec{y}^2 \tag{20.66}$$

for flat space. As we have seen above, in Brinkmann coordinates the metric is manifestly flat,

$$d\bar{s}^2 = 2dudv + d\vec{x}^2 \quad . \tag{20.67}$$

Using the coordinate transformation (20.55) from Rosen to Brinkmann coordinates, we see that the geodesic lines $y^k = c^k$, $V = c$ of the congruence defined by the metric (20.66) correspond to the lines $x^k = c^k u$ in Brinkmann (Minkowski) coordinates. But these are precisely the straight lines through the origin. This explains the coordinate singularity at $U = 0$ and further strengthens the analogy with polar coordinates mentioned at the end of section 20.1.

More generally, we see from (20.55) that the relation between the Brinkmann coordinates x^a and the Rosen coordinates y^k,

$$x^a = \bar{E}^a_k(U) y^k \quad , \tag{20.68}$$

and hence the expression for the geodesic lines $y^k = c^k$, becomes degenerate when $\bar{E}^a_k$ becomes degenerate, i.e. precisely when $\bar{g}_{ij}$ becomes degenerate. Brinkmann coordinates, on the other hand, provide a global coordinate chart for plane wave metrics.

The (almost) inevitablity of (coordinate) singularities in Rosen coordinates can be seen from the following argument.[10] Namely, it follows from the oscillator equation (20.57) that the determinant

$$\mathsf{E} = \det(\bar{E}^a_k) \tag{20.69}$$

satisfies

$$\ddot{\mathsf{E}}/\mathsf{E} = \operatorname{Tr} A + \big((\operatorname{Tr} M)^2 - \operatorname{Tr}(M^2)\big) \leq \operatorname{Tr} A = -\bar{R}_{uu} \quad , \tag{20.70}$$

where use has been made of the expression $\bar{R}_{uu} = -\operatorname{Tr} A$ (20.37) for the Ricci tensor, and where M_{ab} is the symmetric matrix (20.53)

$$M_{ab} = \dot{\bar{E}}_{ai} \bar{E}^i{}_b \quad . \tag{20.71}$$

In particular, therefore, if $\bar{R}_{uu} > 0$, then $\mathsf{E}(u)$ is strictly concave downwards but positive at a non-degenerate point, so that necessarily $\mathsf{e}(u_0) = 0$ for some finite value of u_0, and the Rosen coordinate system breaks down there. By (20.44), $\bar{R}_{uu} > 0$ is equivalent to positivity of the lightcone energy density, a very reasonable requirement on the matter content.

20.7 The Heisenberg Isometry Algebra of a Generic Plane Wave

We now study the isometries of a generic plane wave metric. In Brinkmann coordinates, because of the explicit dependence of the metric on u and the transverse coordinates, only one isometry is manifest, namely that generated by the parallel null vector $Z = \partial_v$. In Rosen coordinates, the metric depends neither on V nor on the transverse coordinates y^k, and one sees that in addition to $Z = \partial_V$ there are at least d more Killing vectors,

[10] This is adapted from G. Gibbons, *Quantized Fields Propagating in Plane-Wave Spacetimes*, Commun. Math. Phys. 45 (1975) 191-202.

namely the ∂_{y^k}. Together these form an Abelian translation algebra acting transitively on the null hypersurfaces of constant U.

However, this is not the whole story. Indeed, one particularly interesting and peculiar feature of plane wave space-times is the fact that they generically possess a *solvable* (rather than semi-simple) isometry algebra, namely a Heisenberg algebra, only part of which we have already seen above.

All Killing vectors V can be found in a systematic way by solving the Killing equations

$$L_V g_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu = 0 \ . \tag{20.72}$$

I will not do this here but simply present the results of this analysis in Brinkmann coordinates. The upshot is that a generic $(2+d)$-dimensional plane wave metric has a $(2d+1)$-dimensional isometry algebra generated by the Killing vector $Z = \partial_v$ and the $2d$ Killing vectors

$$X(f_{(K)}) \equiv X_{(K)} = f_{(K)a}\partial_a - \dot{f}_{(K)a}x^a\partial_v \ . \tag{20.73}$$

Here the $f_{(K)a}$, $K = 1, \ldots, 2d$ are the $2d$ linearly independent solutions of the *harmonic oscillator equation* (again!)

$$\ddot{f}_a(u) = A_{ab}(u)f_b(u) \ . \tag{20.74}$$

These Killing vectors satisfy the algebra

$$[X_{(J)}, X_{(K)}] = W(f_{(J)}, f_{(K)})Z \tag{20.75}$$

$$[X_{(J)}, Z] = 0 \ . \tag{20.76}$$

Here $W(f_{(J)}, f_{(K)})$, the *Wronskian* of the two solutions, is defined by

$$W(f_{(J)}, f_{(K)}) = \sum_a (\dot{f}_{(J)a}f_{(K)a} - \dot{f}_{(K)a}f_{(J)a}) \ . \tag{20.77}$$

It is constant (independent of u) as a consequence of the harmonic oscillator equation. Thus $W(f_{(J)}, f_{(K)})$ is a constant, non-degenerate, even-dimensional antisymmetric matrix (non-degeneracy is implied by the linear independence of the solutions $f_{(J)}$.) Hence it can be put into standard (Darboux) form. Explicitly, a convenient choice of basis for the solutions $f_{(J)}$ is obtained by splitting the $f_{(J)}$ into two sets of solutions

$$\{f_{(J)}\} \rightarrow \{p_{(a)}, q_{(a)}\} \tag{20.78}$$

characterised by the initial conditions

$$\begin{aligned} p_{(a)b}(u_0) &= \delta_{ab} \quad \dot{p}_{(a)b}(u_0) = 0 \\ q_{(a)b}(u_0) &= 0 \quad \dot{q}_{(a)b}(u_0) = \delta_{ab} \ . \end{aligned} \tag{20.79}$$

Since the Wronskian of these functions is independent of u, it can be determined by evaluating it at $u = u_0$. Then one can immediately read off that

$$\begin{aligned} W(q_{(a)}, q_{(b)}) &= W(p_{(a)}, p_{(b)}) = 0 \\ W(q_{(a)}, p_{(b)}) &= \delta_{ab} \ . \end{aligned} \tag{20.80}$$

Therefore the corresponding Killing vectors

$$Q_{(a)} = X(q_{(a)}) \ , \qquad P_{(a)} = X(p_{(a)}) \tag{20.81}$$

and Z satisfy the canonically normalised Heisenberg algebra

$$\begin{aligned} &[Q_{(a)}, Z] = [P_{(a)}, Z] = 0 \\ &[Q_{(a)}, Q_{(b)}] = [P_{(a)}, P_{(b)}] = 0 \\ &[Q_{(a)}, P_{(b)}] = \delta_{ab} Z \ . \end{aligned} \tag{20.82}$$

20.8 Plane Waves with more Isometries

Generically, a plane wave metric has just this Heisenberg algebra of isometries. It acts transitively on the null hyperplanes $u = const.$, with a simply transitive Abelian subalgebra. However, for special choices of $A_{ab}(u)$, there may of course be more Killing vectors. These could arise from internal symmetries of A_{ab}, giving more Killing vectors in the transverse directions. For example, the conformally flat plane waves (20.43) have an additional $SO(d)$ symmetry (and conversely $SO(d)$-invariance implies conformal flatness).

Of more interest to us is the fact that for particular $A_{ab}(u)$ there may be Killing vectors with a ∂_u-component. The existence of such a Killing vector renders the plane wave homogeneous (away form the fixed points of this extra Killing vector). The obvious examples are plane waves with a u-independent profile A_{ab},

$$ds^2 = 2dudv + A_{ab}x^a x^b du^2 + d\vec{x}^2 \ , \tag{20.83}$$

which have the extra Killing vector $X = \partial_u$. Since A_{ab} is u-independent, it can be diagonalised by a u-independent orthogonal transformation acting on the x^a. Moreover, the overall scale of A_{ab} can be changed, $A_{ab} \to \mu^2 A_{ab}$, by the coordinate transformation (boost)

$$(u, v, x^a) \to (\mu u, \mu^{-1} v, x^a) \ . \tag{20.84}$$

Thus these metrics are classified by the eigenvalues of A_{ab} up to an overall scale and permutations of the eigenvalues.

Since A_{ab} is constant, the Riemann curvature tensor is covariantly constant,

$$\bar{\nabla}_\mu \bar{R}_{\lambda\nu\rho\sigma} = 0 \Leftrightarrow \partial_u A_{ab} = 0 \ . \tag{20.85}$$

Thus a plane wave with constant wave profile A_{ab} is what is known as a locally symmetric space.

The existence of the additional Killing vector $X = \partial_u$ extends the Heisenberg algebra to the harmonic oscillator algebra, with X playing the role of the number operator or harmonic oscillator Hamiltonian. Indeed, X and $Z = \partial_v$ obviously commute, and the commutator of X with one of the Killing vectors $X(f)$ is

$$[X, X(f)] = X(\dot{f}) \ . \tag{20.86}$$

Note that this is consistent, i.e. the right-hand-side is again a Killing vector, because when A_{ab} is constant and f satisfies the harmonic oscillator equation then so does its u-derivative $\dot{f}$. In terms of the basis (20.81), we have

$$\begin{aligned} [X, Q_{(a)}] &= P_{(a)} \\ [X, P_{(a)}] &= A_{ab} Q_{(b)} \ , \end{aligned} \tag{20.87}$$

which is the harmonic oscillator algebra.

Another way of understanding the relation between $X = \partial_u$ and the harmonic oscillator Hamiltonian is to look at the conserved charge associated with X for particles moving along geodesics. As we have seen in section 6.6, given any Killing vector X, the quantity

$$Q_X = X_\mu \dot{x}^\mu \tag{20.88}$$

is constant along the trajectory of the geodesic $x^\mu(\tau)$. For $X = \partial_u$ one finds

$$Q_X = p_u = g_{u\mu} \dot{x}^\mu \tag{20.89}$$

which we had already identified (up to a constant for non-null geodesics) as minus the harmonic oscillator Hamiltonian in section 20.3. This is indeed a conserved charge iff the Hamiltonian is time-independent i.e. iff A_{ab} is constant.

We thus see that the dynamics of particles in a symmetric plane wave background is intimately related to the geometry of the background itself.

Another class of examples of plane waves with an interesting additional Killing vector are plane waves with the non-trivial profile

$$A_{ab}(u) = u^{-2} B_{ab} \tag{20.90}$$

for some constant matrix $B_{ab} = A_{ab}(1)$. Without loss of generality one can then assume that B_{ab} and A_{ab} are diagonal, with eigenvalues the oscillator frequency squares $-\omega_a^2$,

$$A_{ab} = -\omega_a^2 \delta_{ab} u^{-2} \ . \tag{20.91}$$

The corrrresponding plane wave metric

$$d\bar{s}^2 = 2dudv + B_{ab} x^a x^b \frac{du^2}{u^2} + d\vec{x}^2 \tag{20.92}$$

is invariant under the boost/scaling (20.84), corresponding to the extra Killing vector

$$X = u\partial_u - v\partial_v \ . \tag{20.93}$$

Note that in this case the Killing vector $Z = \partial_v$ is no longer a central element of the isometry algebra, since it has a non-trivial commutator with X,

$$[X, Z] = Z \quad . \tag{20.94}$$

Moreover, one finds that the commutator of X with a Heisenberg algebra Killing vector $X(f)$, f_a a solution to the harmonic oscillator equation, is the Heisenberg algebra Killing vector

$$[X, X(f)] = X(u\dot{f}) \quad , \tag{20.95}$$

corresponding to the solution $u\dot{f}_a = u\partial_u f_a$ of the harmonic oscillator equation.

This concludes our brief discussion of plane wave metrics even though much more can and perhaps should be said about plane wave and pp-wave metrics, in particular in the context of the so-called *Penrose Limit* construction. For more on this see my lecture notes[11] (from which I also took the material in this chapter).

[11] M. Blau, *Lecture Notes on Plane Waves and Penrose Limits*, available from `http://www.blau.itp.unibe.ch/Lecturenotes.html`

21 * Kaluza-Klein Theory

21.1 Motivation: Gravity and Gauge Theory

Looking at the Einstein equations and the variational principle, we see that gravity is nicely geometrised while the matter part has to be added by hand and is completely non-geometric. This may be perfectly acceptable for phenomenological Lagrangians (like that for a perfect fluid in Cosmology), but it would clearly be desirable to have a unified description of all the fundamental forces of nature.

Today, the fundamental forces of nature are described by two very different concepts. On the one hand, we have - as we have seen - gravity, in which forces are replaced by geometry, and on the other hand there are the gauge theories of the electroweak and strong interactions (the standard model) or their (grand unified, ...) generalisations.

Thus, if one wants to unify these forces with gravity, there are two possibilities:

1. One can try to realise gravity as a gauge theory (and thus geometry as a consequence of the gauge principle).
2. Or one can try to realise gauge theories as gravity (and hence make them purely geometric).

The first is certainly an attractive idea and has attracted a lot of attention. It is also quite natural since, in a broad sense, gravity is already a gauge theory in the sense that it has a local invariance (under general coordinate transformations or, actively, diffeomorphisms). Also, the behaviour of Christoffel symbols under general coordinate transformations is analogous to the transformation behaviour of non-Abelian gauge fields under gauge transformations, and the whole formalism of covariant derivatives and curvatures is reminiscent of that of non-Abelian gauge theories.

At first sight, equating the Christoffel symbols with gauge fields (potentials) may appear to be a bit puzzling because we originally introduced the metric as the potential of the gravitational field and the Christoffel symbol as the corresponding field strength (representing the gravitational force). However, as we know, the concept of 'force' is itself a gauge (coordinate) dependent concept in General Relativity, and therefore these 'field strengths' behave more like gauge potentials themselves, with their curvature, the Riemann curvature tensor, encoding the gauge covariant information about the gravitational field. This fact, which reflects deep properties of gravity not shared by other forces, is just one of many which suggest that an honest gauge theory interpretation of gravity may be hard to come by. But let us proceed in this direction for a little while anyway.

Clearly, the gauge group should now not be some 'internal' symmetry group like $U(1)$ or $SU(3)$, but rather a space-time symmetry group itself. Among the gauge groups that have been suggested in this context, one finds

1. the translation group (this is natural because, as we have seen, the generators of coordinate transformations are infinitesimal translations)
2. the Lorentz group (this is natural if one wants to view the Christoffel symbols as the analogues of the gauge fields of gravity)
3. and the Poincaré group (a combination of the two).

However, what - by and large - these investigations have shown is that the more one tries to make a gauge theory look like Einstein gravity the less it looks like a standard gauge theory and vice versa.

The main source of difference between gauge theory and gravity is the fact that in the case of Yang-Mills theory the internal indices bear no relation to the space-time indices whereas in gravity these are the same - contrast $F^a_{\mu\nu}$ with $(F^\lambda_\sigma)_{\mu\nu} = R^\lambda{}_{\sigma\mu\nu}$.

In particular, in gravity one can contract the 'internal' with the space-time indices to obtain a scalar Lagrangian, R, linear in the curvature tensor. This is fortunate because, from the point of view of the metric, this is already a two-derivative object.

For Yang-Mills theory, on the other hand, this is not possible, and in order to construct a Lagrangian which is a singlet under the gauge group one needs to contract the space-time and internal indices separately, i.e. one has a Lagrangian quadratic in the field stregths. This gives the usual two-derivative action for the gauge potentials.

In spite of these and other differences and difficulties, this approach has not been completely abandoned and the gauge theory point of view is still very fruitful and useful provided that one appreciates the crucial features that set gravity apart from standard gauge theories.

The second possibility alluded to above, to realise gauge theories as gravity, is much more radical. But how on earth is one supposed to achieve this? The crucial idea has been known since 1919/20 (T. Kaluza), with important contributions by O. Klein (1926). So what is this idea?

21.2 The Kaluza-Klein Miracle: History and Overview

In the early parts of the last century, the only other fundamental force that was known, in addition to gravity, was electro-magnetism, In 1919, Kaluza submitted a paper (to Einstein) in which he made a number of remarkable observations.

First of all, he stressed the similarity between Christoffel symbols and the Maxwell field strength tensor,

$$\begin{aligned}\Gamma_{\mu\nu\lambda} &= \tfrac{1}{2}(\partial_\nu g_{\mu\lambda} - \partial_\mu g_{\nu\lambda} + \partial_\lambda g_{\mu\nu}) \\ F_{\nu\mu} &= \partial_\nu A_\mu - \partial_\mu A_\nu \ .\end{aligned} \tag{21.1}$$

He then noted that $F_{\mu\nu}$ looks like a truncated Christoffel symbol and proposed, in order to make this more manifest, to introduce a *fifth dimension* with a metric such that $\Gamma_{\mu\nu 5} \sim F_{\mu\nu}$. This is inded possible. If one makes the identification

$$A_\mu = g_{\mu 5} \ , \tag{21.2}$$

and the assumption that $g_{\mu 5}$ is independent of the fifth coordinate x^5, then one finds, using the standard formula for the Christoffel symbols, now extended to five dimensions, that

$$\begin{aligned}\Gamma_{\mu\nu 5} &= \tfrac{1}{2}(\partial_5 g_{\mu\nu} + \partial_\nu g_{\mu 5} - \partial_\mu g_{\nu 5}) \\ &= \tfrac{1}{2}(\partial_\nu A_\mu - \partial_\mu A_\nu) = \tfrac{1}{2}F_{\nu\mu} \ .\end{aligned} \tag{21.3}$$

But much more than this is true. Kaluza went on to show that when one postulates a five-dimensional metric of the form (hatted quantities will from now on refer to five dimensional quantities)

$$d\hat{s}^2 = g_{\mu\nu}dx^\mu dx^\nu + (dx^5 + A_\mu dx^\mu)^2 \ , \tag{21.4}$$

and calculates the five-dimensional Einstein-Hilbert Lagrangian $\widehat{R}$, one finds *precisely the four-dimensional Einstein-Maxwell Lagrangian*

$$\widehat{R} = R - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \ . \tag{21.5}$$

This fact is affectionately known as the *Kaluza-Klein Miracle*! Moreover, the five-dimensional geodesic equation turns into the four-dimensional Lorentz force equation for a charged particle, and in this sense gravity and Maxwell theory have really been unified in five-dimensional gravity.

However, although this is very nice, rather amazing in fact, and is clearly trying to tell us something deep, there are numerous problems with this and it is not really clear what has been achieved:

1. Should the fifth direction be treated as real or as a mere mathematical device?
2. If it is to be treated as real, why should one make the assumption that the fields are independent of x^5? But if one does not make this assumption, one will not get Einstein-Maxwell theory.
3. Moreover, if the fifth dimension is to be taken seriously, why are we justified in setting $g_{55} = 1$? If we do not do this, we will not get Einstein-Maxwell theory.

4. If the fifth dimension is real, why have we not discovered it yet?

In spite of all this and other questions, related to non-Abelian gauge symmetries or the quantum behaviour of these theories, Kaluza's idea has remained popular ever since or, rather, has periodically created psychological epidemics of frantic activity, interrupted by dormant phases. Today, Kaluza's idea, with its many reincarnations and variations, is an indispensable and fundamental ingredient in the modern theories of theoretical high energy physics (supergravity and string theories) and many of the questions/problems mentioned above have been addressed, understood and overcome.

Let us now look at this more precisely. We consider a five-dimensional space-time with coordinates $\hat{x}^M = (x^\mu, x^5)$ and a metric of the form (21.4). For later convenience, we will introduce a parameter λ into the metric (even though we will set $\lambda = 1$ for the time being) and write it as

$$d\hat{s}^2 = g_{\mu\nu}dx^\mu dx^\nu + (dx^5 + \lambda A_\mu dx^\mu)^2 \quad . \tag{21.6}$$

More explictly, we therefore have

$$\begin{aligned} \hat{g}_{\mu\nu} &= g_{\mu\nu} + A_\mu A_\nu \\ \hat{g}_{\mu 5} &= A_\mu \\ \hat{g}_{55} &= 1 \quad . \end{aligned} \tag{21.7}$$

The determinant of the metric is $\hat{g} = g$, and the inverse metric has components

$$\begin{aligned} \hat{g}^{\mu\nu} &= g^{\mu\nu} \\ \hat{g}^{\mu 5} &= -A^\mu \\ \hat{g}^{55} &= 1 + A_\mu A^\mu \quad . \end{aligned} \tag{21.8}$$

We will (for now) assume that nothing depends on x^5 (in the old Kaluza-Klein literature this assumption is known as the *cylindricity condition*).

Introducing the notation

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ B_{\mu\nu} &= \partial_\mu A_\nu + \partial_\nu A_\mu \quad , \end{aligned} \tag{21.9}$$

the Christoffel symbols are readily found to be

$$\begin{aligned} \hat{\Gamma}^\mu_{\nu\lambda} &= \Gamma^\mu_{\nu\lambda} - \tfrac{1}{2}(F^\mu_{\ \nu} A_\lambda + F^\mu_{\ \lambda} A_\nu) \\ \hat{\Gamma}^5_{\nu\lambda} &= \tfrac{1}{2}B_{\nu\lambda} - \tfrac{1}{2}A^\mu(F_{\nu\mu}A_\lambda + F_{\lambda\mu}A_\nu) - A^\mu\Gamma_{\mu\nu\lambda} \\ \hat{\Gamma}^\mu_{5\lambda} &= -\tfrac{1}{2}F^\mu_{\ \lambda} \\ \hat{\Gamma}^5_{5\mu} &= -\tfrac{1}{2}F_{\mu\nu}A^\nu \\ \hat{\Gamma}^\mu_{55} &= \hat{\Gamma}^5_{55} = 0 \quad . \end{aligned} \tag{21.10}$$

This does not look particularly encouraging, in particular because of the presence of the $B_{\mu\nu}$ term, but Kaluza was not discouraged and proceeded to calculate the Riemann tensor. I will spare you all the components of the Riemann tensor, but the Ricci tensor we need:

$$\begin{aligned}\widehat{R}_{\mu\nu} &= R_{\mu\nu} + \tfrac{1}{2}F_\mu{}^\rho F_{\rho\nu} + \tfrac{1}{4}F^{\lambda\rho}F_{\lambda\rho}A_\mu A_\nu + \tfrac{1}{2}(A_\nu\nabla_\rho F_\mu{}^\rho + A_\mu\nabla_\rho F_\nu{}^\rho)\\ \widehat{R}_{5\mu} &= +\tfrac{1}{2}\nabla_\nu F_\mu{}^\nu + \tfrac{1}{4}A_\mu F_{\nu\lambda}F^{\nu\lambda}\\ \widehat{R}_{55} &= \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\ .\end{aligned} \tag{21.11}$$

This looks a bit more attractive and covariant but still not very promising. [However, if you work in an orthonormal basis, if you know what that means, the result looks much nicer. In such a basis only the first two terms in $\widehat{R}_{\mu\nu}$ and the first term in $\widehat{R}_{5\mu}$ are present and $\widehat{R}_{55}$ is unchanged, so that all the non-covariant looking terms disappear.] Now the miracle happens. Calculating the curvature scalar, all the annoying terms drop out and one finds

$$\widehat{R} = R - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\ , \tag{21.12}$$

i.e. the Lagrangian of Einstein-Maxwell theory. For $\lambda \neq 1$, the second term would have been multiplied by λ^2. We now consider the five-dimensional pure gravity Einstein-Hilbert action

$$\widehat{S} = \frac{1}{8\pi\widehat{G}}\int\sqrt{\widehat{g}}d^5x\ \widehat{R}\ . \tag{21.13}$$

In order for the integral over x^5 to converge we assume that the x^5-direction is a circle with radius L and we obtain

$$\widehat{S} = \frac{2\pi L}{8\pi\widehat{G}}\int\sqrt{g}d^4x\ (R - \tfrac{1}{4}\lambda^2F_{\mu\nu}F^{\mu\nu})\ . \tag{21.14}$$

Therefore, if we make the identifications

$$\begin{aligned}G &= \widehat{G}/2\pi L\\ \lambda^2 &= 8\pi G\ ,\end{aligned} \tag{21.15}$$

we obtain

$$\widehat{S} = \frac{1}{8\pi G}\int\sqrt{g}d^4x\ R - \frac{1}{4}\int\sqrt{g}d^4x\ F_{\mu\nu}F^{\mu\nu}\ , \tag{21.16}$$

i.e. precisely the four-dimensional Einstein-Maxwell Lagrangian! This amazing fact, that coupled gravity gauge theory systems can arise from higher-dimensional pure gravity, is certainly trying to tell us something.

21.3 The Origin of Gauge Invariance

In physics, at least, miracles require a rational explanation. So let us try to understand on a priori grounds why the Kaluza-Klein miracle occurs. For this, let us recall Kaluza's ansatz for the line element (21.4),

$$d\widehat{s}^2_{KK} = g_{\mu\nu}(x^\lambda)dx^\mu dx^\nu + (dx^5 + A_\mu(x^\lambda)dx^\mu)^2\ . \tag{21.17}$$

and contrast this with the most general form of the line element in five dimensions, namely

$$\begin{aligned} d\widehat{s}^2 &= \widehat{g}_{MN}(x^L)dx^M dx^N \\ &= \widehat{g}_{\mu\nu}(x^\lambda, x^5)dx^\mu dx^\nu + 2\widehat{g}_{\mu 5}(x^\lambda, x^5)dx^\mu dx^5 + \widehat{g}_{55}(x^\mu, x^5)(dx^5)^2 \ . \end{aligned} \tag{21.18}$$

Clearly, the form of the general five-dimensional line element (21.18) is invariant under arbitrary five-dimensional general coordinate transformations $x^M \to \xi^{M'}(x^N)$. This is not true, however, for the Kaluza-Klein ansatz (21.17), as a general x^5-dependent coordinate transformation would destroy the x^5-independence of $\widehat{g}_{\mu\nu} = g_{\mu\nu}$ and $\widehat{g}_{\mu 5} = A_\mu$ and would also not leave $\widehat{g}_{55} = 1$ invariant.

The form of the Kaluza-Klein line element is, however, invariant under the following two classes of coordinate transformations:

1. There are four-dimensional coordinate transformations

   $$\begin{aligned} x^5 &\to x^5 \\ x^\mu &\to \xi^{\nu'}(x^\mu) \end{aligned} \tag{21.19}$$

   Under these transformations, as we know, $g_{\mu\nu}$ transforms in such a way that $g_{\mu\nu}dx^\mu dx^\nu$ is invariant, $A_\mu = \widehat{g}_{\mu 5}$ transforms as a four-dimensional covector, thus $A_\mu dx^\mu$ is invariant, and the whole metric is invariant.

2. There is also another remnant of five-dimensional general covariance, namely

   $$\begin{aligned} x^5 &\to \xi^5(x^\mu, x^5) = x^5 + f(x^\mu) \\ x^\mu &\to \xi^\mu(x^\nu) = x^\mu \ . \end{aligned} \tag{21.20}$$

   Under this transformation, $g_{\mu\nu}$ and g_{55} are invariant, but $A_\mu = g_{\mu 5}$ changes as

   $$\begin{aligned} A'_\mu &= \widehat{g}'_{\mu 5} = \frac{\partial x^M}{\partial \xi^\mu}\frac{\partial x^N}{\partial \xi^5}\widehat{g}_{MN} \\ &= \frac{\partial x^M}{\partial x^\mu} g_{\mu 5} \\ &= g_{\mu 5} - \frac{\partial f}{\partial x^\mu} g_{55} \\ &= A_\mu - \partial_\mu f \ . \end{aligned} \tag{21.21}$$

   In other words, the Kaluza-Klein line element is invariant under the shift $x^5 \to x^5 + f(x^\mu)$ accompanied by $A_\mu \to A_\mu - \partial_\mu f$ (and this can of course also be read off directly from the metric).

But this is precisely a *gauge transformation* of the vector potential A_μ and we see that in the present context *gauge transformations arise as remnants of five-dimensional general covariance*!

But now it is clear that we are guaranteed to get Einstein-Maxwell theory in four dimensions: First of all, upon integration over x^5, the shift in x^5 is irrelevant and starting with the five-dimensional Einstein-Hilbert action we are bound to end up with an action in four dimensions, depending on $g_{\mu\nu}$ and A_μ, which is (a) generally covariant (in the four-dimensional sense), (b) second order in derivatives, and (c) invariant under gauge transformations of A_μ. But then the only possibility is the Einstein-Maxwell action.

A fruitful way of looking at the origin of this gauge invariance is as a consequence of the fact that constant shifts in x^5 are isometries of the metric, i.e. that $\partial/\partial x^5$ is a Killing vector of the metric (21.17). Then the isometry group of the 'internal' circle in the x^5-direction, namely $SO(2)$, becomes the gauge group $U(1) = SO(2)$ of the four-dimensional theory.

From this point of view, the gauge transformation of the vector potential arises from the Lie derivative of $\widehat{g}_{\mu 5}$ along the vector field $f(x^\mu)\partial_5$:

$$
\begin{aligned}
Y = f(x^\mu)\partial_5 \quad &\Rightarrow \quad Y^\mu = 0 \\
& \quad Y^5 = f \\
&\Rightarrow \quad Y_\mu = A_\mu f \\
& \quad Y_5 = f \quad .
\end{aligned}
\tag{21.22}
$$

$$
\begin{aligned}
(L_Y\widehat{g})_{\mu 5} &= \widehat{\nabla}_\mu Y_5 + \widehat{\nabla}_5 Y_\mu \\
&= \partial_\mu Y_5 - 2\widehat{\Gamma}^\mu{}_{5M} Y^M \\
&= \partial_\mu f + F^\nu_{\ \mu} Y_\nu + F_{\mu\nu} A^\nu Y_5 \\
&= \partial_\mu f \\
&\Leftrightarrow \quad \delta A_\mu = -\partial_\mu f \quad .
\end{aligned}
\tag{21.23}
$$

This point of view becomes particularly useful when one wants to obtain non-Abelian gauge symmetries in this way (via a Kaluza-Klein reduction): One starts with a higher-dimensional internal space with isometry group G and makes an analogous ansatz for the metric. Then among the remnants of the higher-dimensional general coordinate transformations there are, in particular, x^μ-dependent 'isometries' of the internal metric. These act like non-Abelian gauge transformations on the off-block-diagonal components of the metric and, upon integration over the internal space, one is guaranteed to get, perhaps among other things, the four-dimensional Einstein-Hilbert and Yang-Mills actions.

21.4 Geodesics

There is something else that works very beautifully in this context, namely the description of the motion of charged particles in four dimensions moving under the combined influence of a gravitational and an electro-magnetic field. As we will see, also these two effects are unfied from a five-dimensional Kaluza-Klein point of view.

Let us consider the five-dimensional geodesic equation

$$\ddot{x}^M + \widehat{\Gamma}^M_{NL}\dot{x}^N\dot{x}^L = 0 \ . \tag{21.24}$$

Either because the metric (and hence the Lagrangian) does not depend on x^5, or because we know that $V = \partial_5$ is a Killing vector of the metric, we know that we have a conserved quantity

$$\frac{\partial\mathcal{L}}{\partial\dot{x}^5} \sim V_M\dot{x}^M = \dot{x}^5 + A_\mu\dot{x}^\mu \ , \tag{21.25}$$

along the geodesic world lines. We will see in a moment what this quantity corresponds to. The remaining x^μ-component of the geodesic equation is

$$\begin{aligned}\ddot{x}^\mu + \widehat{\Gamma}^\mu_{NL}\dot{x}^N\dot{x}^L &= \ddot{x}^\mu + \widehat{\Gamma}^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda \\ &+ 2\widehat{\Gamma}^\mu_{\nu 5}\dot{x}^\nu\dot{x}^5 + 2\widehat{\Gamma}^\mu_{55}\dot{x}^5\dot{x}^5 \\ &= \ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda - F^\mu_\nu A_\lambda\dot{x}^\nu\dot{x}^\lambda - F^\mu_\nu\dot{x}^\nu\dot{x}^5 \\ &= \ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda - F^\mu_\nu\dot{x}^\nu(A_\lambda\dot{x}^\lambda + \dot{x}^5) \ .\end{aligned} \tag{21.26}$$

Therefore this component of the geodesic equation is equivalent to

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = (A_\lambda\dot{x}^\lambda + \dot{x}^5)F^\mu_\nu\dot{x}^\nu \ . \tag{21.27}$$

This is precisely the Lorentz law if one identifies the constant of motion with the ratio of the charge and the mass of the particle,

$$\dot{x}^5 + A_\mu\dot{x}^\mu = \frac{e}{m} \ . \tag{21.28}$$

Hence electro-magnetic and gravitational forces are indeed unified. The fact that charged particles take a different trajectory from neutral ones is not a violation of the equivalence principle but only reflects the fact that they started out with a different velocity in the x^5-direction!

21.5 First Problems: The Equations of Motion

The equations of motion of the four-dimensional Einstein-Hilbert-Maxwell action will of course give us the coupled Einstein-Maxwell equations

$$\begin{aligned}R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R &= 8\pi G T_{\mu\nu} \\ \nabla_\mu F^{\mu\nu} &= 0 \ .\end{aligned} \tag{21.29}$$

But now let us take a look at the equations of motion following from the five-dimensional Einstein-Hilbert action. These are, as we are looking at the vacuum equations, just the Ricci-flatness equations $\widehat{R}_{MN} = 0$. But looking back at (21.11) we see that these are clearly *not* equivalent to the Einstein-Maxwell equations. In particular, $\widehat{R}_{55} = 0$ imposes the constraint

$$\widehat{R}_{55} = 0 \quad \Rightarrow \quad F_{\mu\nu} F^{\mu\nu} = 0 \quad , \tag{21.30}$$

and only then do the remaining equations $\widehat{R}_{\mu\nu} = 0$, $\widehat{R}_{\mu 5} = 0$ become equivalent to the Einstein-Maxwell equations (21.29).

What happened? Well, for one, taking variations and making a particular ansatz for the field configurations in the variational principle are two operations that in general do not commute. In particular, the Kaluza-Klein ansatz is special because it imposes the condition $g_{55} = 1$. Thus in four dimensions there is no equation of motion corresponding to $\widehat{g}_{55}$ whereas $\widehat{R}_{55} = 0$, the additional constraint, is just that, the equation arising from varying $\widehat{g}_{55}$. Thus Einstein-Maxwell theory is not a consistent truncation of five-dimensional General Relativity.

But now we really have to ask ourselves what we have actually achieved. We would like to claim that the five-dimensional Einstein-Hilbert action unifies the four-dimensional Einstein-Hilbert and Maxwell actions, but on the other hand we want to reject the five-dimensional Einstein equations? But then we are not ascribing any dynamics to the fifth dimension and are treating the Kaluza-Klein miracle as a mere kinematical, or mathematical, or bookkeeping device for the four-dimensional fields. This is clearly rather artificial and unsatisfactory.

There are some other unsatisfactory features as well in the theory we have developed so far. For instance we demanded that there be no dependence on x^5, which again makes the five-dimensional point of view look rather artificial. If one wants to take the fifth dimension seriously, one has to allow for an x^5-dependence of all the fields (and then explain later, perhaps, why we have not yet discovered the fifth dimension in every-day or high energy experiments).

21.6 Masses and Charges from Scalar Fields in Five Dimenions

With these issues in mind, we will now revisit the Kaluza-Klein ansatz, regarding the fifth dimension as real and exploring the consequences of this. Instead of considering directly the effect of a full (i.e. not restricted by any special ansatz for the metric) five-dimensional metric on four-dimensional physics, we will start with the simpler case of a free massless scalar field in five dimensions.

Let us assume that we have a five-dimensional space-time of the form $M_5 = M_4 \times S^1$ where we will at first assume that M_4 is Minkowski space and the metric is simply

$$d\hat{s}^2 = -dt^2 + d\vec{x}^2 + (dx^5)^2 \quad , \tag{21.31}$$

with x^5 a coordinate on a circle with radius L. Now consider a massless scalar field $\widehat{\phi}$

on M_5, satisfying the five-dimensional massless Klein-Gordon equation

$$\widehat{\Box}\widehat{\phi}(x^\mu, x^5) = \widehat{\eta}^{MN}\partial_M\partial_N\widehat{\phi}(x^\mu, x^5) = 0 \ . \tag{21.32}$$

As x^5 is periodic with period $2\pi L$, we can make a Fourier expansion of $\widehat{\phi}$ to make the x^5-dependence more explicit,

$$\widehat{\phi}(x^\mu, x^5) = \sum_n \phi_n(x^\mu)\mathrm{e}^{inx^5/L} \ . \tag{21.33}$$

Plugging this expansion into the five-dimensional Klein-Gordon equation, we find that this turns into an infinite number of decoupled equations, one for each Fourier mode of ϕ_n of $\widehat{\phi}$, namely

$$(\Box - m_n^2)\phi_n = 0 \ . \tag{21.34}$$

Here $\Box$ of course now refers to the four-dimensional d'Alembertian, and the mass term

$$m_n^2 = \frac{n^2}{L^2} \tag{21.35}$$

arises from the x^5-derivative ∂_5^2 in $\widehat{\Box}$.

Thus we see that, from a four-dimensional perspective, a massless scalar field in five dimensions give rise to one massless scalar field in four dimensions (the harmonic or constant mode on the internal space) and an infinite number of massive fields. The masses of these fields, known as the Kaluza-Klein modes, have the behaviour $m_n \sim n/L$. In general, this behaviour, an infinite tower of massive fields with mass $\sim 1/$ length scale is characteristic of massive fields arising from dimensional reduction from some higher dimensional space.

Next, instead of looking at a scalar field on Minkowski space times a circle with the product metric, let us consider the Kaluza-Klein metric,

$$d\widehat{s}^2 = -dt^2 + d\vec{x}^2 + (dx^5 + \lambda A_\mu dx^\mu)^2 \ , \tag{21.36}$$

and the corresponding Klein-Gordon equation

$$\widehat{\Box}\widehat{\phi}(x^\mu, x^5) = \widehat{g}^{MN}\widehat{\nabla}_M\partial_N\widehat{\phi}(x^\mu, x^5) = 0 \ . \tag{21.37}$$

Rather than spelling this out in terms of Christoffel symbols, it is more convenient to use (4.49) and recall that $\sqrt{\widehat{g}} = \sqrt{g} = 1$ to write this as

$$\begin{aligned}
\widehat{\Box} &= \partial_M(\widehat{g}^{MN}\partial_N) \\
&= \partial_\mu\widehat{g}^{\mu\nu}\partial_\nu + \partial_5\widehat{g}^{5\mu}\partial_\mu + \partial_\mu\widehat{g}^{\mu 5}\partial_5 + \partial_5\widehat{g}^{55}\partial_5 \\
&= \eta^{\mu\nu}\partial_\mu\partial_\nu + \partial_5(-\lambda A^\mu\partial_\mu) + \partial_\mu(-\lambda A^\mu\partial_5) + (1 + \lambda^2 A_\mu A^\mu)\partial_5\partial_5 \\
&= \eta^{\mu\nu}(\partial_\mu - \lambda A_\mu\partial_5)(\partial_\nu - \lambda A_\nu\partial_5) + (\partial_5)^2 \ .
\end{aligned} \tag{21.38}$$

Acting with this operator on the Fourier decomposition of $\widehat{\phi}$, we evidently again get an infinte number of decoupled equations, one for each Fourier mode ϕ_n of $\widehat{\phi}$, namely

$$\left[\eta^{\mu\nu}(\partial_\mu - i\frac{\lambda n}{L}A_\mu)(\partial_\nu - i\frac{\lambda n}{L}A_\nu) - m_n^2\right]\phi_n = 0 \ . \tag{21.39}$$

This shows that the non-constant ($n \neq 0$) modes are not only massive but also charged under the gauge field A_μ. Comparing the operator

$$\partial_\mu - i\frac{\lambda n}{L}A_\mu \tag{21.40}$$

with the standard form of the minimal coupling,

$$\frac{\hbar}{i}\partial_\mu - eA_\mu \ , \tag{21.41}$$

we learn that the electric charge e_n of the n'th mode is given by

$$\frac{e_n}{\hbar} = \frac{n\lambda}{L} \ . \tag{21.42}$$

In particular, these charges are all integer multiples of a basic charge, $e_n = ne$, with

$$e = \frac{\hbar\lambda}{L} = \frac{\sqrt{8\pi G}\hbar}{L} \ . \tag{21.43}$$

Thus we get a formula for L, the radius of the fifth dimension,

$$L^2 = \frac{8\pi G\hbar^2}{e^2} = \frac{8\pi G\hbar}{e^2/\hbar} \ . \tag{21.44}$$

Restoring the velocity of light in this formula, and identifying the present $U(1)$ gauge symmetry with the standard gauge symmetry, we recognise here the fine structure constant

$$\alpha = e^2/4\pi\hbar c \approx 1/137 \ , \tag{21.45}$$

and the *Planck length*

$$\ell_P = \sqrt{\frac{G\hbar}{c^3}} \approx 10^{-33}\text{cm} \ . \tag{21.46}$$

Thus

$$L^2 = \frac{2\ell_P^2}{\alpha} \approx 274\ell_P^2 \ . \tag{21.47}$$

This is very small indeed, and it is therefore no surprise that this fifth dimension, if it is the origin of the $U(1)$ gauge invariance of the world we live in, has not yet been seen.

Another way of saying this is that the fact that L is so tiny implies that the masses m_n are huge, not far from the *Planck mass*

$$m_P = \sqrt{\frac{\hbar c}{G}} \approx 10^{-5}\text{g} \approx 10^{19}\text{GeV} \ . \tag{21.48}$$

These would never have been spotted in present-day accelerators. Thus the massive modes are completely irrelevant for low-energy physics, the non-constant modes can be dropped, and this provides a justification for neglecting the x^5-dependence. However, this also means that the charged particles we know (electrons, protons, ...) cannot possibly be identified with these Kaluza-Klein modes.

The way modern Kaluza-Klein theories address this problem is by identifying the light charged particles we observe with the massless Kaluza-Klein modes. One then requires the standard spontaneous symmetry breaking mechanism to equip them with the small masses required by observation. This still leaves the question of how these particles should pick up a charge (as the zero modes are not only massless but also not charged). This is solved by going to higher dimensions, with non-Abelian gauge groups, for which massless particles are no longer necessarily singlets of the gauge group (they could e.g. live in the adjoint).

21.7 Kinematics of Dimensional Reduction

We have seen above that a massless scalar field in five dimensions gives rise to a massless scalar field plus an infinite tower of massive scalar fields in four dimensions. What happens for other fields (after all, we are ultimately interested in what happens to the five-dimensional metric)?

Consider, for example, a five-dimensional vector potential (covector field) $\widehat{B}_M(x^N)$. From a four-dimensional vantage point this looks like a four-dimensional vector field $B_\mu(x^\nu, x^5)$ and a scalar $\phi(x^\mu, x^5) = B_5(x^\mu, x^5)$. Fourier expanding, one will then obtain in four dimensions:

1. one massless Abelian gauge field $B_\mu(x^\nu)$
2. an infinite tower of massive charged vector fields
3. one massless scalar field $\phi(x^\mu) = B_5(x^\mu)$
4. an infinite tower of massive charged scalar fields

Retaining, for the same reasons as before, only the massless, i.e. x^5-independent, modes we therefore obtain a theory involving one scalar field and one Abelian vector field from pure Maxwell theory in five dimensions. The Lagrangian for these fields would be (dropping all x^5-derivatives)

$$\begin{aligned} F_{MN}F^{MN} &= F_{\mu\nu}F^{\mu\nu} + 2F_{\mu 5}F^{\mu 5} \\ &\to F_{\mu\nu}F^{\mu\nu} + 2(\partial_\mu \phi)(\partial^\mu \phi) \ . \end{aligned} \tag{21.49}$$

This procedure of obtaining Lagrangians in lower dimensions from Lagrangians in higher dimensions by simply dropping the dependence on the 'internal' coordinates is known as dimensional reduction or Kaluza-Klein reduction. But the terminology is not uniform here - sometimes the latter term is used to indicate the reduction including all the massive modes. Also, in general 'massless' is not the same as 'x^5-independent', and then Kaluza-Klein reduction may refer to keeping the massless modes rather than the x^5-independent modes one retains in dimensional reduction.

Likewise, we can now consider what happens to the five-dimensional metric $\widehat{g}_{MN}(x^L)$. From a four-dimensional perspective, this splits into three different kinds of fields, namely a symmetric tensor $\widehat{g}_{\mu\nu}$, a covector $A_\mu = \widehat{g}_{\mu 5}$ and a scalar $\phi = \widehat{g}_{55}$. As before, these will each give rise to a massless field in four dimensions (which we interpret as the metric, a vector potential and a scalar field) as well as an infinite number of massive fields.

We see that, in addition to the massless fields we considered before, in the old Kaluza-Klein ansatz, we obtain one more massless field, namely the scalar field ϕ. Thus, even if we may be justified in dropping all the massive modes, we should keep this massless field in the ansatz for the metric and the action. With this in mind we now return to the Kaluza-Klein ansatz.

21.8 The Kaluza-Klein Ansatz Revisited

Let us once again consider pure gravity in five dimensions, i.e. the Einstein-Hilbert action

$$\widehat{S} = \frac{1}{8\pi\widehat{G}} \int \sqrt{\widehat{g}} d^5x \; \widehat{R} \;\; . \tag{21.50}$$

Let us now parametrise the full five-dimensional metric as

$$d\widehat{s}^2 = \phi^{-1/3}[g_{\mu\nu}dx^\mu dx^\mu + \phi(dx^5 + \lambda A_\mu dx^\mu)^2] \;\; , \tag{21.51}$$

where all the fields depend on all the coordinates x^μ, x^5. Any five-dimensional metric can be written in this way and we can simply think of this as a change of variables

$$\widehat{g}_{MN} \to (g_{\mu\nu}, A_\mu, \phi) \;\; . \tag{21.52}$$

In matrix form, this metric reads

$$(\widehat{g}_{MN}) = \phi^{-1/3} \begin{pmatrix} g_{\mu\nu} + \lambda^2\phi A_\mu A_\nu & \lambda\phi A_\nu \\ \lambda\phi A_\mu & \phi \end{pmatrix} \tag{21.53}$$

For a variety of reasons, this particular parametrisation is useful. In particular, it reduces to the Kaluza-Klein ansatz when $\phi = 1$ and all the fields are independent of x^5 and the ϕ's in the off-diagonal component ensure that the determinant of the metric is independent of the A_μ.

The only thing that may require some explanation is the strange overall power of ϕ. To see why this is a good choice, assume that the overall power is ϕ^a for some a. Then for $\sqrt{\widehat{g}}$ one finds

$$\sqrt{\widehat{g}} = \phi^{5a/2}\phi^{1/2}\sqrt{g} = \phi^{(5a+1)/2}\sqrt{g} \ . \tag{21.54}$$

On the other hand, for the Ricci tensor one has, schematically,

$$\hat{R}_{\mu\nu} = R_{\mu\nu} + \dots \ , \tag{21.55}$$

and therefore

$$\begin{aligned}\hat{R} &= \widehat{g}^{\mu\nu}R_{\mu\nu} + \dots \\ &= \phi^{-a}g^{\mu\nu}R_{\mu\nu} + \dots \\ &= \phi^{-a}R + \dots \ .\end{aligned} \tag{21.56}$$

Hence the five-dimensional Einstein-Hilbert action reduces to

$$\begin{aligned}\sqrt{\widehat{g}}\widehat{R} &\sim \phi^{(5a+1)/2}\phi^{-a}\sqrt{g}R + \dots \\ &= \phi^{(3a+1)/2}\sqrt{g}R + \dots \ .\end{aligned} \tag{21.57}$$

Thus, if one wants the five-dimensional Einstein-Hilbert action to reduce to the standard four-dimensional Einstein-Hilbert action (plus other things), without any non-minimal coupling of the scalar field ϕ to the metric, one needs to choose $a = -1/3$ which is the choice made in (21.51,21.53).

Making a Fourier-mode expansion of all the fields, plugging this into the Einstein-Hilbert action

$$\frac{1}{8\pi\widehat{G}}\int\sqrt{\widehat{g}}d^5x\ \hat{R} \ , \tag{21.58}$$

integrating over x^5 and retaining only the constant modes $g_{(0)\mu\nu}$, $A_{(0)\mu}$ and $\phi_{(0)}$, one obtains the action

$$S = \int\sqrt{g}d^4x\ \left[\frac{1}{8\pi G}R(g_{(0)\mu\nu}) - \frac{1}{4}\phi_{(0)}F_{(0)\mu\nu}F^{\mu\nu}_{(0)} - \frac{1}{48\pi G}\phi^{-2}_{(0)}g^{\mu\nu}_{(0)}\partial_\mu\phi_{(0)}\partial_\nu\phi_{(0)}\right] \ . \tag{21.59}$$

Here we have once again made the identifications (21.15). This action may not look as nice as before, but it is what it is. It is at least generally covariant and gauge invariant, as expected. We also see very clearly that it is inconsistenst with the equations of motion for $\phi_{(0)}$,

$$\Box\log\phi_{(0)} = \frac{3}{4}8\pi G\phi_{(0)}F_{(0)\mu\nu}F^{\mu\nu}_{(0)} \ , \tag{21.60}$$

to set $\phi_{(0)} = 1$ as this would imply $F_{(0)\mu\nu}F^{\mu\nu}_{(0)} = 0$, in agreement with our earlier observations regarding $\hat{R}_{55} = 0$.

However, the configuration $g_{(0)\mu\nu} = \eta_{\mu\nu}$, $A_{(0)\mu} = 0$, $\phi_{(0)} = 1$ is a solution to the equations of motion and defines the 'vacuum' or ground state of the theory. From this point of view the zero mode metric, (21.53) with the fields replaced by their zero modes, i.e. the Kaluza-Klein ansatz with the inclusion of ϕ, has the following interpretation: as usual in quantum theory, once one has chosen a vacuum, one can consider fluctuations around that vacuum. The fields $g_{(0)\mu\nu}, A_{(0)\mu}, \phi_{(0)}$ are then the massless fluctuations around the vacuum and are the fields of the low-energy action. The full classical or quantum theory will also contain all the massive and charged Kaluza-Klein modes.

21.9 Non-Abelian Generalisation and Outlook

Even though in certain respects the Abelian theory we have discussed above is atypical, it is rather straightforward to generalise the previous considerations from Maxwell theory to Yang-Mills theory for an arbitrary non-Abelian gauge group. Of course, to achieve that, one needs to consider higher-dimensional internal spaces, i.e. gravity in $4+d$ dimensions, with a space-time of the form $M_4 \times M_d$. The crucial observation is that gauge symmetries in four dimensions arise from isometries (Killing vectors) of the metric on M_d.

Let the coordinates on M_d be x^a, denote by g_{ab} the metric on M_d, and let K_i^a, $i = 1, \ldots, n$ denote the n linearly independent Killing vectors of the metric g_{ab}. These generate the Lie algebra of the isometry group G via the Lie bracket

$$[K_i, K_j]^a \equiv K_i^b \partial_b K_j^a - K_j^b \partial_b K_i^a = f_{ij}^k K_k^a \ . \tag{21.61}$$

M_d could for example be the group manifold of the Lie group G itself, or a homogeneous space G/H for some subgroup $H \subset G$.

Now consider the following Kaluza-Klein ansatz for the metric,

$$d\hat{s}^2 = g_{\mu\nu} dx^\mu dx^\nu + g_{ab}(dx^a + K_i^a A_\mu^i dx^\mu)(dx^b + K_j^b A_\nu^j dx^\nu) \ . \tag{21.62}$$

Note the appearance of fields with the correct index structure to act as non-Abelian gauge fields for the gauge group G, namely the A_μ^i. Again these should be thought of as fluctuations of the metric around its 'ground state', $M_4 \times M_d$ with its product metric $(g_{\mu\nu}, g_{ab})$.

Now consider an infinitesimal coordinate transformation generated by the vector field

$$V^a(x^\mu, x^b) = f^i(x^\mu) K_i^a(x^b) \ , \tag{21.63}$$

i.e.

$$\delta x^a = f^i(x^\mu) K_i^a(x^b) \ . \tag{21.64}$$

This leaves the form of the metric invariant, and

$$\delta \hat{g}_{\mu a} = L_V \hat{g}_{\mu a} \tag{21.65}$$

can be seen to imply

$$\delta A^i_\mu = D_\mu f^i \equiv \partial_\mu f^i - f^i_{jk} A^j_\mu f^k \quad , \tag{21.66}$$

i.e. precisely an infinitesimal non-Abelian gauge transformation. The easiest way to see this is to use the form of the Lie derivative not in its covariant form,

$$L_V \widehat{g}_{\mu a} = \widehat{\nabla}_\mu V_a + \widehat{\nabla}_a V_\mu \tag{21.67}$$

(which requires knowledge of the Christoffel symbols) but in the form

$$L_V \widehat{g}_{\mu a} = V^c \partial_c \widehat{g}_{\mu a} + \partial_\mu V^c \widehat{g}_{ca} + \partial_a V^c \widehat{g}_{\mu c} \quad . \tag{21.68}$$

Inserting the definitions of $\widehat{g}_{\mu a}$ and V^a, using the fact that the K^a_i are Killing vectors of the metric g_{ab} and the relation (21.61), one finds

$$L_V \widehat{g}_{\mu a} = g_{ab} K^b_i D_\mu f^i \quad , \tag{21.69}$$

and hence (21.66).

One is then assured to find a Yang-Mills like term

$$L_{YM} \sim F^i_{\mu\nu} F^{j\,\mu\nu} K^a_i K^b_j g_{ab} \tag{21.70}$$

in the reduction of the Lagrangian from $4+d$ to 4 dimensions.

The problem with this scenario (already prior to worrying about the inclusion of scalar fields, of which there will be plenty in this case, one for each component of g_{ab}) is that the four-dimensional space-time cannot be chosen to be flat. Rather, it must have a huge cosmological constant. This arises because the dimensional reduction of the $(4+d)$-dimensional Einstein-Hilbert Lagrangian $\hat{R}$ will also include a contribution from the scalar curvature R_d of the metric on M_d. For a compact internal space with non-Abelian isometries this scalar curvature is non-zero and will therefore lead to an effective cosmological constant in the four-dimensional action. This cosmological constant could be cancelled 'by hand' by introducing an appropriate cosmological constant of the opposite sign into the $(d+4)$-dimensional Einstein-Hilbert action, but this looks rather contrived and artificial.

Nevertheless, this and other problems have not stopped people from looking for 'realistic' Kaluza-Klein theories giving rise to the standard model gauge group in four dimension. Of course, in order to get the standard model action or something resembling it, fermions need to be added to the $(d+4)$-dimensional action.

An interesting observation in this regard is that the lowest possible dimension for a homogenous space with isometry group $G = SU(3) \times SU(2) \times U(1)$ is seven, so that the dimension of space-time is eleven. This arises because the maximal compact subgroup H of G, giving rise to the smallest dimensional homogeneous space G/H of G, is $SU(2) \times U(1) \times U(1)$. As the dimension of G is $8+3+1=12$ and that of H is $3+1+1=5$, the dimension of G/H is $12 - 5 = 7$. This is intriguing because eleven is also the highest dimension in which supergravity exists (in higher dimensions, supersymmetry would require the existence of spin > 2 particles). That, plus the hope that supergravity would have a better quantum behaviour than ordinary gravity, led to an enourmous amount of activity on Kaluza-Klein supergravity in the early 80's.

Unfortunately, it turned out that not only was supergravity sick at the quantum level as well but also that it is impossible to get a chiral fermion spectrum in four dimensions from pure gravity plus spinors in $(4+d)$ dimensions. One way around the latter problem is to include explicit Yang-Mills fields already in $(d+4)$-dimensions, but that appeared to defy the purpose of the whole Kaluza-Klein idea.

Today, the picture has changed and supergravity is regarded as a low-energy approximation to string theory which is believed to give a consistent description of quantum gravity. These string theories typically live in ten dimensions, and thus one needs to 'compactify' the theory on a small internal six-dimensional space, much as in the Kaluza-Klein idea. Even though non-Abelian gauge fields now typically do not arise from Kaluza-Klein reduction but rather from explicit gauge fields in ten dimensions (or objects called *D-branes*), in all other respects Kaluza's old idea is alive, doing very well, and an indispensable part of the toolkit of modern theoretical high energy physics.

THE END