Entanglement of Four-Qubit Rank-2 Mixed States

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Abstract

It is known that there are three maximally entangled states $|\Phi_1\rangle = (|0000\rangle + |1111\rangle)/\sqrt{2}$, $|\Phi_2\rangle = (\sqrt{2}|1111\rangle + |1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle)/\sqrt{6}$, and $|\Phi_3\rangle = (|1111\rangle + |1100\rangle + |0010\rangle + |0001\rangle)/2$ in four-qubit system. It is also known that there are three independent measures $\mathcal{F}_j^{(4)}$ $(j = 1, 2, 3)$ for true four-way quantum entanglement in the same system. In this paper we compute $\mathcal{F}_j^{(4)}$ and their corresponding linear monotones $\mathcal{G}_j^{(4)}$ for three rank-two mixed states $\rho_j = p|\Phi_j\rangle\langle\Phi_j| + (1 - p)|\mathrm{W}_4\rangle\langle\mathrm{W}_4|$, where $|\mathrm{W}_4\rangle = (|0111\rangle + |1011\rangle + |1101\rangle + |1110\rangle)/2$. we discuss the possible applications of our results briefly.

I. INTRODUCTION

Recently, much attention is being paid to quantum information theory (QIT) and quantum technology (QT)[1]. Most important notion in QIT and QT is a quantum correlation, which is usually termed by entanglement[2] of given quantum states. As shown for last two decades it plays a central role in quantum teleportation[3], superdense coding[4], quantum cloning[5], and quantum cryptography[6, 7]. It is also quantum entanglement, which makes the quantum computer[1] outperform the classical one[9]. Thus, it is very important to understand how to quantify and how to characterize the entanglement.

For bipartite quantum system many entanglement measures were constructed before such as distillable entanglement[10], entanglement of formation (EoF)[10], and relative entropy of entanglement (REE)[11, 12]. Especially, for two-qubit system, EoF is expressed as[13]

$$\mathcal{E}(C) = h\left(\frac{1+\sqrt{1-C^2}}{2}\right), \tag{1.1}$$

where $h(x)$ is a binary entropy function $h(x) = -x\ln x - (1-x)\ln(1-x)$ and C is called the concurrence. For two-qubit pure state $|\psi\rangle = \psi_{ij}|ij\rangle$ with $(i, j = 0, 1)$, C is given by

$$C = |\epsilon_{i_1 i_2}\epsilon_{j_1 j_2}\psi_{i_1 j_1}\psi_{i_2 j_2}| = 2|\psi_{00}\psi_{11} - \psi_{01}\psi_{10}|, \tag{1.2}$$

where the Einstein convention is understood and $\epsilon_{\mu\nu}$ is an antisymmetric tensor. For two-qubit mixed state ρ the concurrence $C(\rho)$ can be computed by $C = \max(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0)$, where $\{\lambda_1^2, \lambda_2^2, \lambda_3^2, \lambda_4^2\}$ are eigenvalues of $\rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$ with increasing order. Thus, one can compute the EoF for all two-qubit states in principle. Still, however, the closed formulae for distillable entanglement and REE were not found even if many strategies were developed in Ref.[14] and Ref.[15], respectively.

Although quantification of the entanglement is important, it is equally important to classify the entanglement, i.e., to classify the quantum states into the same type of entanglement. The most popular classification scheme is a classification through a stochastic local operation and classical communication (SLOCC)[16]. If $|\psi\rangle$ and $|\phi\rangle$ are in same SLOCC class, this means that $|\psi\rangle$ and $|\phi\rangle$ can be used to implement same task of quantum information process although the probability of success for this task is different. Mathematically, if two

[1] The current status of quantum computer technology was reviewed in Ref.[8].

n-party states $|\psi\rangle$ and $|\phi\rangle$ are in the same SLOCC class, they are related to each other by $|\psi\rangle = A_1 \otimes A_2 \otimes \cdots \otimes A_n|\phi\rangle$ with $\{A_j\}$ being arbitrary invertible local operators[2]. However, it is more useful to restrict ourselves to SLOCC transformation where all $\{A_j\}$ belong to SL(2, C), the group of 2×2 complex matrices having determinant equal to 1. In the three-qubit pure-state system it was shown[17] that there are six different SLOCC classes, fully-separable, three bi-separable, W, and Greenberger-Horne-Zeilinger (GHZ) classes. Subsequently, the classification was extended to three-qubit mixed-state system[18].

Classification through the SLOCC transformation enables us to construct the entanglement measures for the multipartite states. As Ref.[19] showed, any linearly homogeneous positive function of a pure state that is invariant under determinant 1 SLOCC operations is an entanglement monotone. One can show that the concurrence C in Eq. (1.2) is such an entanglement monotone as follows. Let $|\psi\rangle = \psi_{ij}|ij\rangle$ with $i,j = 0,1$. Then, $|\tilde{\psi}\rangle \equiv (A \otimes B)|\psi\rangle = \tilde{\psi}_{ij}|ij\rangle$, where $\tilde{\psi}_{ij} = \psi_{\alpha\beta}A_{i\alpha}B_{j\beta}$. Using $\epsilon_{ij}M_{i\alpha}M_{j\beta} = (\det M)\epsilon_{\alpha\beta}$ for arbitrary matrix M, it is easy to show $\epsilon_{i_1 i_2}\epsilon_{j_1 j_2}\tilde{\psi}_{i_1 j_1}\tilde{\psi}_{i_2 j_2} = (\det A)(\det B)\epsilon_{i_1 i_2}\epsilon_{j_1 j_2}\psi_{i_1 j_1}\psi_{i_2 j_2}$, which implies that C is invariant under determinant 1 SLOCC operations.

The theorem in Ref.[19] can be applied to the three-qubit system. If $|\psi\rangle = \psi_{ijk}|ijk\rangle$, the invariant monotone is

$$\tau_3 = \left| 2\epsilon_{i_1 i_2}\epsilon_{i_3 i_4}\epsilon_{j_1 j_2}\epsilon_{j_3 j_4}\epsilon_{k_1 k_3}\epsilon_{k_2 k_4}\psi_{i_1 j_1 k_1}\psi_{i_2 j_2 k_2}\psi_{i_3 j_3 k_3}\psi_{i_4 j_4 k_4} \right|^{1/2}. \tag{1.3}$$

This is exactly the same with a square root of the residual entanglement[3] introduced in Ref.[20]. The three-tangle (1.3) has following properties. If $|\psi\rangle$ is a fully-separable or a partially-separable state, its three-tangle completely vanishes. Thus, τ_3 measures the true three-way entanglement. It also gives $\tau_3(\mathrm{GHZ}_3) = 1$ and $\tau_3(\mathrm{W}_3) = 0$ to the three-way entangled states, where

$$|\mathrm{GHZ}_3\rangle \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \qquad\qquad |\mathrm{W}_3\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle). \tag{1.4}$$

For mixed state quantification of the entanglement is usually defined via a convex-roof method[10, 21]. Although the concurrence for an arbitrary two-qubit mixed state can be, in principle, computed following the procedure introduced in Ref.[13], still we do not know how to compute the three-tangle (or residual entanglement) for an arbitrary three-qubit mixed

[2] For complete proof on the connection between SLOCC and local operations see Appendix A of Ref.[17].
[3] In this paper we will call τ_3 three-tangle and τ_3^2 residual entanglement.

state. However, the residual entanglement for several special mixtures were computed in Ref.[22]. More recently, the three-tangle for all GHZ-symmetric states[23] was computed analytically[24].

It is also possible to construct the SLOCC-invariant monotones in the higher-qubit systems. In the higher-qubit systems, however, there are many independent monotones, because the number of independent SLOCC-invariant monotones is equal to the degrees of freedom of pure quantum state minus the degrees of freedom induced by the determinant 1 SLOCC operations. For example, there are $2(2^n - 1) - 6n$ independent monotones in n-qubit system. Thus, there are six invariant monotones in four-qubit system. Among them, it was shown in Ref.[25] by making use of the antilinearity[21] that there are following three independent monotones which measure the true four-way entanglement:

$$\begin{aligned}
\mathcal{F}_1^{(4)} &= (\sigma_\mu \sigma_\nu \sigma_2 \sigma_2) \bullet (\sigma^\mu \sigma_2 \sigma_\lambda \sigma_2) \bullet (\sigma_2 \sigma^\nu \sigma^\lambda \sigma_2) \\
\mathcal{F}_2^{(4)} &= (\sigma_\mu \sigma_\nu \sigma_2 \sigma_2) \bullet (\sigma^\mu \sigma_2 \sigma_\lambda \sigma_2) \bullet (\sigma_2 \sigma^\nu \sigma_2 \sigma_\tau) \bullet (\sigma_2 \sigma_2 \sigma^\lambda \sigma^\tau) \\
\mathcal{F}_3^{(4)} &= \frac{1}{2} (\sigma_\mu \sigma_\nu \sigma_2 \sigma_2) \bullet (\sigma^\mu \sigma^\nu \sigma_2 \sigma_2) \bullet (\sigma_\rho \sigma_2 \sigma_\tau \sigma_2) \bullet (\sigma^\rho \sigma_2 \sigma^\tau \sigma_2) \bullet (\sigma_\kappa \sigma_2 \sigma_2 \sigma_\lambda) \bullet (\sigma^\kappa \sigma_2 \sigma_2 \sigma^\lambda),
\end{aligned} \tag{1.5}$$

where $\sigma_1 = \mathbb{1}_2$, $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, $\sigma_3 = \sigma_z$, and the Einstein convention is introduced with a metric $g^{\mu\nu} = \mathrm{diag}\{-1, 1, 0, 1\}$. Furthermore, it was shown in Ref.[26] that there are following three maximally entangled states in four-qubit system:

$$\begin{aligned}
|\Phi_1\rangle &= \frac{1}{\sqrt{2}} \big(|0000\rangle + |1111\rangle \big) \\
|\Phi_2\rangle &= \frac{1}{\sqrt{6}} \left(\sqrt{2} |1111\rangle + |1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle \right) \\
|\Phi_3\rangle &= \frac{1}{2} \big(|1111\rangle + |1100\rangle + |0010\rangle + |0001\rangle \big).
\end{aligned} \tag{1.6}$$

	$\mathcal{F}_1^{(4)}$	$\mathcal{F}_2^{(4)}$	$\mathcal{F}_3^{(4)}$
$\lvert\Phi_1\rangle$	1	1	$\frac{1}{2}$
$\lvert\Phi_2\rangle$	$\frac{8}{9}$	0	0
$\lvert\Phi_3\rangle$	0	0	1
$\lvert \mathrm{W}_4\rangle$	0	0	0

Table I:$\mathcal{F}_1^{(4)}$, $\mathcal{F}_2^{(4)}$, and $\mathcal{F}_3^{(4)}$ of the maximally entangled and W_4 states.

The measures $\mathcal{F}_1^{(4)}$, $\mathcal{F}_2^{(4)}$, and $\mathcal{F}_3^{(4)}$ of $|\Phi_1\rangle$, $|\Phi_2\rangle$, $|\Phi_3\rangle$, and

$$|\mathrm{W}_4\rangle = \frac{1}{2} \big(|0111\rangle + |1011\rangle + |1101\rangle + |1110\rangle \big) \tag{1.7}$$

are summarized in Table I. As Table I shows, $|\Phi_1\rangle$ is detected by all measures while $|\Phi_2\rangle$ (or $|\Phi_3\rangle$) is detected by only $\mathcal{F}_1^{(4)}$ (or $\mathcal{F}_3^{(4)}$). As three-qubit system, $|\mathrm{W}_4\rangle$ is not detected by all measures.

The purpose of this paper is to compute $\mathcal{F}_j^{(4)}$ and $\mathcal{G}_j^{(4)}$ $(j = 1, 2, 3)$ for the rank-two mixed states $\rho_j = p|\Phi_j\rangle\langle\Phi_j| + (1-p)|\mathrm{W}_4\rangle\langle\mathrm{W}_4|$ $(j = 1, 2, 3)$, where $\mathcal{G}_j^{(4)}$ is a linear entanglement monotone defined as

$$\mathcal{G}_1^{(4)} = \left(\mathcal{F}_1^{(4)}\right)^{1/3} \qquad \mathcal{G}_2^{(4)} = \left(\mathcal{F}_2^{(4)}\right)^{1/4} \qquad \mathcal{G}_3^{(4)} = \left(\mathcal{F}_3^{(4)}\right)^{1/6}. \tag{1.8}$$

In terms of the SLOCC-language $|\Phi_j\rangle$ belong to G_{abcd} and $|\mathrm{W}_4\rangle$ belongs to Lab_3 [27]. Thus, ρ_j are the mixtures of two different SLOCC classes. In this paper we want to understand how the four-qubit entanglement is evolved when a state is moved from one SLOCC-class to the other one.

The paper is organized as follows. In sections II, III, and IV we derive the entanglement of ρ_1, ρ_2, and ρ_3 analytically. We also derive the optimal decompositions explicitly for each range in p. To check the correctness of our results we use the criterion discussed in Ref.[28], i.e. *entanglement should be a convex hull of the minimum of the characteristic curves.* In section V we discuss the possible applications of our results.

II. ENTANGLEMENT OF ρ_1

j	$\mathcal{F}_j^{(4)}$	$\mathcal{G}_j^{(4)}$	p_0
$j = 1$	$p(6p - 2p^2 - 3)\theta(p - p_0)$	$\frac{p-p_0}{1-p_0}\theta(p - p_0)$	$\frac{\sqrt{3}}{\sqrt{3}+1} \approx 0.634$
$j = 2$	$p^2[p^2 - 4(1-p)^2]\theta(p - p_0)$	$\frac{p-p_0}{1-p_0}\theta(p - p_0)$	$\frac{2}{3} \approx 0.667$
$j = 3$	$\frac{p^6}{2}$	$\frac{p}{2^{1/6}}$	

Table II:Summary of $\mathcal{F}_j^{(4)}$ and $\mathcal{G}_j^{(4)}$ for ρ_1

In this section we will compute the entanglement of $\rho_1 = p|\Phi_1\rangle\langle\Phi_1| + (1-p)|\mathrm{W}_4\rangle\langle\mathrm{W}_4|$. We start with three-qubit pure state

$$|Z_1(p, \varphi)\rangle = \sqrt{p}|\Phi_1\rangle - e^{i\varphi}\sqrt{1-p}|\mathrm{W}_4\rangle. \tag{2.1}$$

Then, one can show

$$\begin{aligned}
\mathcal{F}_1^{(4)}\left[Z_1(p,\varphi)\right] &= p|p^2 - 3(1-p)^2 e^{4i\varphi}| \\
\mathcal{F}_2^{(4)}\left[Z_1(p,\varphi)\right] &= p^2|p^2 - 4(1-p)^2 e^{4i\varphi}| \\
\mathcal{F}_3^{(4)}\left[Z_1(p,\varphi)\right] &= \frac{p^6}{2}.
\end{aligned} \tag{2.2}$$

A. $\mathcal{F}_1^{(4)}(\rho_1)$ and $\mathcal{G}_1^{(4)}(\rho_1)$

From Eq. (2.2) one can show that $\mathcal{F}_1^{(4)}\left[Z_1(p,\varphi)\right]$ has a nontrivial zero ($\varphi = 0$)

$$p_0 = \frac{\sqrt{3}}{\sqrt{3}+1} \approx 0.634. \tag{2.3}$$

The existence of finite p_0 guarantees that $\mathcal{F}_1^{(4)}(\rho_1)$ should vanish at $0 \leq p \leq p_0$. At $p = p_0$ this fact can be verified because we have the optimal decomposition

$$\begin{aligned}
\rho_1(p_0) = \frac{1}{4}\Bigg[&|Z_1(p_0,0)\rangle\langle Z_1(p_0,0)| + |Z_1\left(p_0,\frac{\pi}{2}\right)\rangle\langle Z_1\left(p_0,\frac{\pi}{2}\right)| \\
&+ |Z_1(p_0,\pi)\rangle\langle Z_1(p_0,\pi)| + |Z_1\left(p_0,\frac{3\pi}{2}\right)\rangle\langle Z_1\left(p_0,\frac{3\pi}{2}\right)| \Bigg].
\end{aligned} \tag{2.4}$$

At the region $0 \leq p < p_0$, $\mathcal{F}_1^{(4)}(\rho_1)$ should vanish too because one can find the following optimal decomposition

$$\rho_1(p) = \frac{p}{p_0}\rho_1(p_0) + \left(1 - \frac{p}{p_0}\right)|\mathrm{W}_4\rangle\langle\mathrm{W}_4|. \tag{2.5}$$

Combining these facts, one can conclude that $\mathcal{F}_1^{(4)}(\rho_1) = 0$ at $0 \leq p \leq p_0$.

Next, we consider the $p_0 \leq p \leq 1$ region. Eq. (2.4) at $p = p_0$ strongly suggests that the optimal decomposition at this region is

$$\begin{aligned}
\rho_1(p) = \frac{1}{4}\Bigg[&|Z_1(p,0)\rangle\langle Z_1(p,0)| + |Z_1\left(p,\frac{\pi}{2}\right)\rangle\langle Z_1\left(p,\frac{\pi}{2}\right)| \\
&+ |Z_1(p,\pi)\rangle\langle Z_1(p,\pi)| + |Z_1\left(p,\frac{3\pi}{2}\right)\rangle\langle Z_1\left(p,\frac{3\pi}{2}\right)| \Bigg].
\end{aligned} \tag{2.6}$$

If Eq. (2.6) is a correct optimal decomposition in this region, $\mathcal{F}_1^{(4)}(\rho_1)$ reduces to

$$\mathcal{F}_1^{(4)}(\rho_1) = p(6p - 2p^2 - 3). \tag{2.7}$$

Since the right-hand side of Eq. (2.7) is convex, our conjecture (Eq. (2.6)) seems to be right. In conclusion, we can write

$$\mathcal{F}_1^{(4)}(\rho_1) = \theta(p - p_0)p(6p - 2p^2 - 3), \tag{2.8}$$

where $\theta(x)$ is a step function defined as

$$\theta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0. \end{cases} \tag{2.9}$$

However, if our choice Eq. (2.6) is incorrect, Eq. (2.8) is merely the upper bound of $\mathcal{F}_1^{(4)}(\rho_1)$. Thus, we need to prove that Eq. (2.8) is really optimal value. To prove this one can adopt numerical analysis with few pure state ensembles as Caratheodory's theorem implies. In this paper, however, we will adopt the alternative method presented in Ref.[28]. We plot the p-dependence of $\mathcal{F}_1^{(4)}[Z_1(p,\varphi)]$ for various φ (See solid lines of Fig. 1(a)). These curves have been referred as the characteristic curves. As Ref.[28] showed, $\mathcal{F}_1^{(4)}(\rho_1)$ is a convex hull of the minimum of the characteristic curves. Fig. 1(a) indicates that Eq.(2.8) (thick dashed line) is really the convex characteristic curve, which implies that Eq.(2.8) is really optimal.

Now, let us consider $\mathcal{G}_1^{(4)}(\rho_1)$. It is easy to show that $\mathcal{G}_1^{(4)}(\rho_1)$ vanishes at $0 \leq p \leq p_0$ due to the optimal decomposition Eq. (2.5). If one chooses Eq. (2.6) as an optimal decomposition at $p_0 \leq p \leq 1$, the resulting $\mathcal{G}_1^{(4)}(\rho_1)$ is not convex in the full range. Thus, we should adopt a technique introduced in Ref.[22]. In this case the optimal decomposition is

$$\rho_1(p) = \frac{p-p_0}{1-p_0}|\Phi_1\rangle\langle\Phi_1| + \frac{1-p}{1-p_0}\rho_1(p_0), \tag{2.10}$$

which results in $\mathcal{G}_1^{(4)}(\rho_1) = (p-p_0)/(1-p_0)$. Combining all these facts, one can conclude

$$\mathcal{G}_1^{(4)}(\rho_1) = \theta(p-p_0)\frac{p-p_0}{1-p_0}. \tag{2.11}$$

To confirm that Eq. (2.11) is correct, we plot the characteristic curves $\mathcal{G}_1^{(4)}[Z_1(p,\varphi)]$ for various φ as solid lines and Eq. (2.11) as thick dashed line in Fig. 1(c). This figure shows that Eq. (2.11) is convex hull of the minimum of the characteristic curves, which strongly supports the validity of Eq. (2.11).

B. $\mathcal{F}_2^{(4)}(\rho_1)$ and $\mathcal{G}_2^{(4)}(\rho_1)$

From Eq. (2.2) one can notice that $\mathcal{F}_2^{(4)}[Z_1(p,\varphi)]$ has a nontrivial zero ($\varphi = 0$)

$$p_0 = \frac{2}{3} \approx 0.667. \tag{2.12}$$

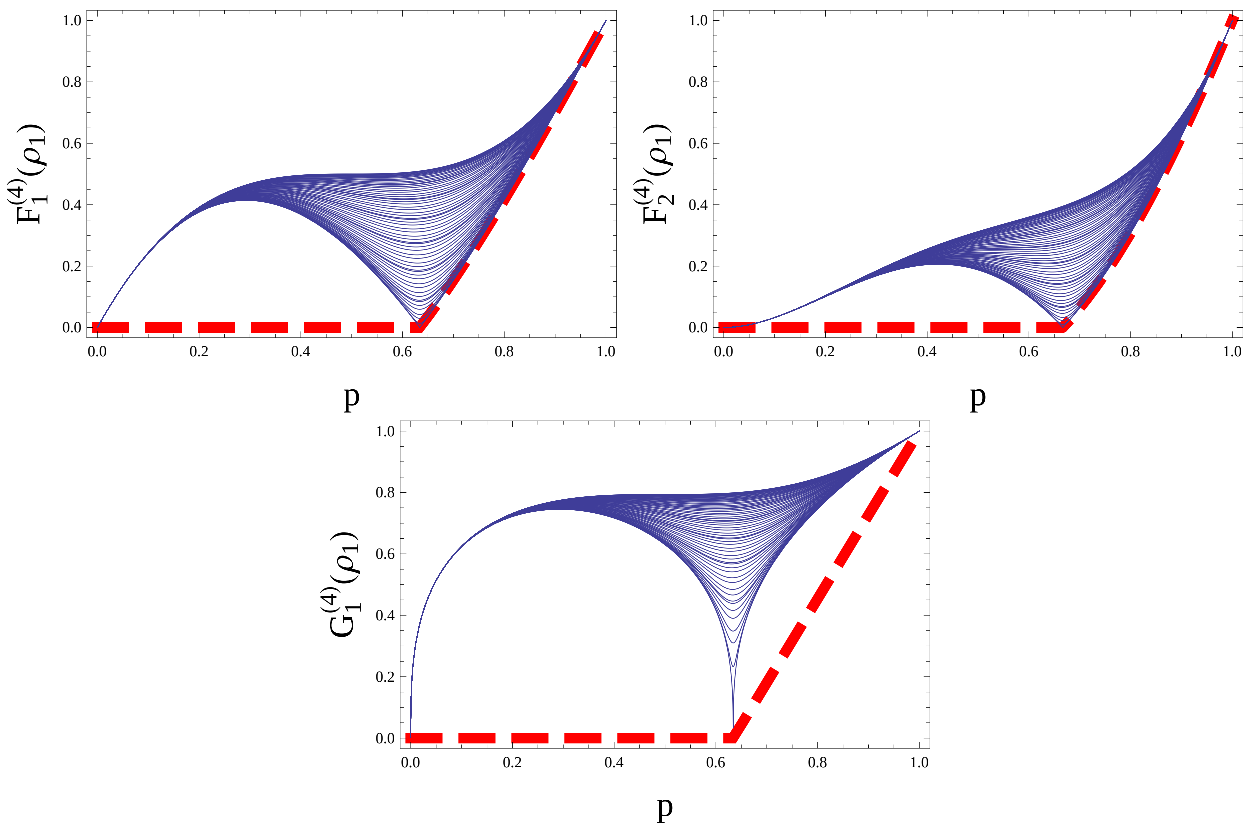

FIG. 1: (Color online) Plot of the p dependence of (a) $\mathcal{F}_1^{(4)}[Z_1(p,\varphi)]$, (b) $\mathcal{F}_2^{(4)}[Z_1(p,\varphi)]$, and (c) $\mathcal{G}_1^{(4)}[Z_1(p,\varphi)]$ in Eq. (2.2). We have chosen φ from 0 to 2π as an interval 0.1. The thick dashed lines correspond to $\mathcal{F}_1^{(4)}(\rho_1)$ in Eq. (2.8), $\mathcal{F}_2^{(4)}(\rho_1)$ in Eq. (2.13) and $\mathcal{G}_1^{(4)}(\rho_1)$ in Eq. (2.11). These figures indicate that Eq. (2.8), Eq. (2.13), and Eq. (2.11) are convex hull of the minimum of the characteristic curves.

Thus, Eq. (2.5) and Eq. (2.6) with $p_0 = 2/3$ can be the optimal decompositions for $\mathcal{F}_2^{(4)}(\rho_1)$ at $0 \leq p \leq p_0$ and $p_0 \leq p \leq 1$, respectively. Then, the resulting $\mathcal{F}_2^{(4)}(\rho_1)$ becomes

$$\mathcal{F}_2^{(4)}(\rho_1) = \theta(p - p_0)p^2[p^2 - 4(1-p)^2]. \tag{2.13}$$

In order to confirm that our result (2.13) is correct, we plot the characteristic curves for various φ (solid lines) and Eq. (2.13) (thick dashed line) in Fig. 1 (b). As Fig. 1(b) exhibits, our result (2.13) is convex hull of the minimum of the characteristic curves, which strongly supports that Eq. (2.13) is really optimal one.

Similarly, $\mathcal{G}_2^{(4)}(\rho_1)$ becomes Eq. (2.11) with changing only p_0 to 2/3. The corresponding optimal decompositions are Eq. (2.5) at $0 \leq p \leq p_0$ and Eq. (2.10) at $p_0 \leq p \leq 1$, respectively. Of course, we have to change p_0 to $2/3$.

C. $\mathcal{F}_3^{(4)}(\rho_1)$ and $\mathcal{G}_3^{(4)}(\rho_1)$

Eq. (2.2) shows that $\mathcal{F}_3^{(4)}[Z_1(p,\varphi)]$ doe not have nontrivial zero. In addition, it is independent of the phase angle φ. This fact may indicate that there are infinite number of optimal decompositions for $\mathcal{F}_3^{(4)}(\rho_1)$. The simplest one is

$$\rho_1(p) = \frac{1}{2}|Z_1(p,0)\rangle\langle Z_1(p,0)| + \frac{1}{2}|Z_1(p,\pi)\rangle\langle Z_1(p,\pi)|, \tag{2.14}$$

which gives $\mathcal{F}_3^{(4)}(\rho_1) = p^6/2$. If one chooses Eq. (2.14) as an optimal decomposition for $\mathcal{G}_3^{(4)}(\rho_1)$, it generates $\mathcal{G}_3^{(4)}(\rho_1) = p/2^{1/6}$. Since it is not concave, we do not need to adopt a technique to make $\mathcal{G}_3^{(4)}(\rho_1)$ convex as we did previously.

III. ENTANGLEMENT OF ρ_2

In this section we would like to quantify the entanglement of ρ_2. Above all, we should say that Table I implies

$$\mathcal{F}_2^{(4)}(\rho_2) = \mathcal{G}_2^{(4)}(\rho_2) = \mathcal{F}_3^{(4)}(\rho_2) = \mathcal{G}_3^{(4)}(\rho_2) = 0, \tag{3.1}$$

because $\rho_2 = p|\Phi_2\rangle\langle\Phi_2| + (1-p)|\mathrm{W}_4\rangle\langle\mathrm{W}_4|$ itself is an optimal decomposition for those entanglement measures. This fact is due to the fact that $\mathcal{F}_2^{(4)}$ and $\mathcal{F}_3^{(4)}$ cannot detect both $|\Phi_2\rangle$ and $|\mathrm{W}_4\rangle$.

Let us now compute $\mathcal{F}_1^{(4)}(\rho_2)$ and $\mathcal{G}_1^{(4)}(\rho_2)$. If we define

$$|Z_2(p,\varphi)\rangle = \sqrt{p}|\Phi_2\rangle - e^{i\varphi}\sqrt{1-p}|\mathrm{W}_4\rangle, \tag{3.2}$$

it is straightforward to show

$$\mathcal{F}_1^{(4)}[Z_2(p,\varphi)] = \frac{8}{9}p^{3/2}|p^{3/2} - 2\sqrt{6}(1-p)^{3/2}e^{3i\varphi}|. \tag{3.3}$$

We notice that $\mathcal{F}_1^{(4)}[Z_2(p,\varphi)]$ has a nontrivial zero ($\varphi = 0$)

$$p_0 = \frac{(2\sqrt{6})^{2/3}}{1+(2\sqrt{6})^{2/3}} \approx 0.743. \tag{3.4}$$

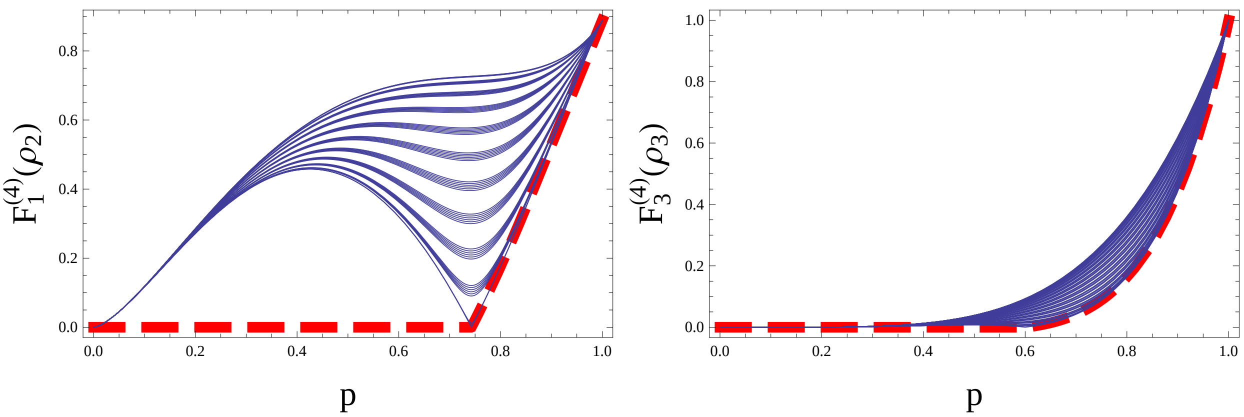

FIG. 2: (Color online) Plot of the p dependence of (a) $\mathcal{F}_1^{(4)}[Z_2(p,\varphi)]$ in Eq. (3.3) and (b) $\mathcal{F}_3^{(4)}[Z_3(p,\varphi)]$ in Eq. (4.3). We have chosen φ from 0 to 2π as an interval 0.1. The thick dashed lines correspond to $\mathcal{F}_1^{(4)}(\rho_2)$ in Eq. (3.12) and $\mathcal{F}_3^{(4)}(\rho_3)$ in Eq. (4.8). These figures indicate that Eq. (3.12) and Eq. (4.8) are convex hull of the minimum of the characteristic curves.

Thus, $\mathcal{F}_1^{(4)}(\rho_2)$ vanishes at $0 \leq p \leq p_0$ because one can fine the optimal decomposition

$$\rho_2(p) = \frac{p}{p_0}\rho_2(p_0) + \left(1 - \frac{p}{p_0}\right)|\mathrm{W}_4\rangle\langle\mathrm{W}_4|, \tag{3.5}$$

where

$$\rho_2(p_0) = \frac{1}{3}\left[|Z_2(p_0,0)\rangle\langle Z_2(p_0,0)| + |Z_2\left(p_0,\frac{2\pi}{3}\right)\rangle\langle Z_2\left(p_0,\frac{2\pi}{3}\right)| + |Z_2\left(p_0,\frac{4\pi}{3}\right)\rangle\langle Z_2\left(p_0,\frac{4\pi}{3}\right)|\right]. \tag{3.6}$$

As the previous cases, we adopt, as a trial, the optimal decomposition at $p_0 \leq p \leq 1$ as

$$\rho_2(p) = \frac{1}{3}\left[|Z_2(p,0)\rangle\langle Z_2(p,0)| + |Z_2\left(p,\frac{2\pi}{3}\right)\rangle\langle Z_2\left(p,\frac{2\pi}{3}\right)| + |Z_2\left(p,\frac{4\pi}{3}\right)\rangle\langle Z_2\left(p,\frac{4\pi}{3}\right)|\right]. \tag{3.7}$$

Then $\mathcal{F}_1^{(4)}(\rho_2)$ becomes $g_I(p)$, where

$$g_I(p) = \frac{8}{9}p^{3/2}\left[p^{3/2} - 2\sqrt{6}(1-p)^{3/2}\right]. \tag{3.8}$$

However, $g_I(p)$ is not convex at the region $p \geq p_* \approx 0.9196$. Thus, we should adopt the technique previously used again to make $g_I(p)$ convex at the large-p region.

Now, we define p_1 such as $p_0 \leq p_1 \leq p_*$. The parameter p_1 will be determined later. At the region $p_1 \leq p \leq 1$ we adopt the optimal decomposition for $\mathcal{F}_1^{(4)}(\rho_2)$ as a following form:

$$\rho_2(p) = \frac{p - p_1}{1 - p_1}|\Phi_2\rangle\langle\Phi_2| + \frac{1-p}{1-p_1}\rho_2(p_1), \tag{3.9}$$

where

$$\rho_2(p_1) = \frac{1}{3}\left[|Z_2(p_1,0)\rangle\langle Z_2(p_1,0)| + |Z_2\left(p_1,\frac{2\pi}{3}\right)\rangle\langle Z_2\left(p_1,\frac{2\pi}{3}\right)| + |Z_2\left(p_1,\frac{4\pi}{3}\right)\rangle\langle Z_2\left(p_1,\frac{4\pi}{3}\right)|\right]. \tag{3.10}$$

Eq. (3.9) leads $\mathcal{F}_1^{(4)}(\rho_2)$ to $g_{II}(p)$ at the large-p region, where

$$g_{II}(p) = \frac{8}{9}\left[\frac{p-p_1}{1-p_1} + \frac{1-p}{1-p_1}\left\{p_1^3 - 2\sqrt{6}p_1^{3/2}(1-p_1)^{3/2}\right\}\right]. \tag{3.11}$$

As expected $g_{II}(p)$ is convex at $p_1 \leq p \leq 1$. The parameter p_1 is determined by $\frac{\partial g_{II}}{\partial p_1} = 0$, which yields $p_1 \approx 0.861$[4]. Thus, $\mathcal{F}_1^{(4)}(\rho_2)$ can be summarized as

$$\mathcal{F}_1^{(4)}(\rho_2) = \begin{cases} 0 & 0 \leq p \leq p_0 \\ g_I(p) & p_0 \leq p \leq p_1 \\ g_{II}(p) & p_1 \leq p \leq 1. \end{cases} \tag{3.12}$$

In order to confirm again that Eq. (3.12) is correct, we plot the p-dependence of the characteristic curves (solid lines) in Fig. 2(a) for various φ. Our result (3.12) is plotted as a thick dashed line. This figure shows that our result (3.12) is a convex characteristic curve, which strongly supports that our result (3.12) is correct.

Now, let us compute $\mathcal{G}_1^{(4)}(\rho_2)$. At $0 \leq p \leq p_0$, $\mathcal{G}_1^{(4)}(\rho_2)$ should be zero due to Eq. (3.5). If we adopt Eq. (3.7) as an optimal decomposition $\mathcal{G}_1^{(4)}(\rho_2) = g_I^{1/3}(p)$ is obtained. However, it is not convex in the full range. Therefore, we have to choose

$$\rho_2(p) = \frac{p-p_0}{1-p_0}|\Phi_2\rangle\langle\Phi_2| + \frac{1-p}{1-p_0}\rho_2(p_0) \tag{3.13}$$

as an optimal decomposition, which results in

$$\mathcal{G}_1^{(4)}(\rho_2) = \theta(p-p_0)\left(\frac{8}{9}\right)^{1/3}\frac{p-p_0}{1-p_0}. \tag{3.14}$$

IV. ENTANGLEMENT OF ρ_3

In this section we will compute the entanglement of $\rho_3 = p|\Phi_3\rangle\langle\Phi_3| + (1-p)|\mathrm{W}_4\rangle\langle\mathrm{W}_4|$. Since $\mathcal{F}_1^{(4)}$ and $\mathcal{F}_2^{(4)}$ cannot detect both $|\Phi_3\rangle$ and $|\mathrm{W}_4\rangle$, it is easy to show

$$\mathcal{F}_1^{(4)}(\rho_3) = \mathcal{G}_1^{(4)}(\rho_3) = \mathcal{F}_2^{(4)}(\rho_3) = \mathcal{G}_2^{(4)}(\rho_3) = 0. \tag{4.1}$$

[4] The parameter p_1 is obtained by an equation $6p_1(4p_1-3)^2 = (1-p_1)(1+2p_1)^2$.

Now, let us compute $\mathcal{F}_3^{(4)}(\rho_3)$ and $\mathcal{G}_3^{(4)}(\rho_3)$. If we define

$$|Z_3(p,\varphi)\rangle = \sqrt{p}|\Phi_3\rangle - e^{i\varphi}\sqrt{1-p}|\mathrm{W}_4\rangle, \tag{4.2}$$

it is possible to show that $\mathcal{F}_3^{(4)}[Z_3(p,\varphi)]$ reduces to

$$\mathcal{F}_3^{(4)}[Z_3(p,\varphi)] = p^5\left|p - \frac{3}{2}(1-p)e^{2i\varphi}\right|. \tag{4.3}$$

Eq. (4.3) implies that $\mathcal{F}_3^{(4)}[Z_3(p,\varphi)]$ has a nontrivial zero ($\varphi = 0$)

$$p_0 = \frac{3}{5} = 0.6. \tag{4.4}$$

Thus, $\mathcal{F}_3^{(4)}(\rho_3)$ should be zero at the region $0 \leq p \leq p_0$ and its optimal decomposition is

$$\rho_3(p) = \frac{p}{p_0}\rho_3(p_0) + \left(1 - \frac{p}{p_0}\right)|\mathrm{W}_4\rangle\langle\mathrm{W}_4|, \tag{4.5}$$

where

$$\rho_3(p_0) = \frac{1}{2}\left[|Z_3(p_0,0)\rangle\langle Z_3(p_0,0)| + |Z_3(p_0,\pi)\rangle\langle Z_3(p_0,\pi)|\right]. \tag{4.6}$$

If we adopt the optimal decomposition at $p_0 \leq p \leq 1$ as a form

$$\rho_3(p) = \frac{1}{2}\left[|Z_3(p,0)\rangle\langle Z_3(p,0)| + |Z_3(p,\pi)\rangle\langle Z_3(p,\pi)|\right], \tag{4.7}$$

the resulting $\mathcal{F}_3^{(4)}(\rho_3)$ becomes $\frac{5}{2}p^5\left(p - \frac{3}{5}\right)$. Since this is convex, we conclude

$$\mathcal{F}_3^{(4)}(\rho_3) = \theta(p - p_0)\frac{5}{2}p^5\left(p - \frac{3}{5}\right). \tag{4.8}$$

In order to prove that Eq. (4.8) is correct we plot again the characteristic curves (solid lines) and our result (4.8) (thick dashed line) in Fig. 2(b), which supports that Eq. (4.8) is optimal one.

Finally, let us compute $\mathcal{G}_3^{(4)}(\rho_3)$. If we take Eq. (4.7) as an optimal decomposition for $\mathcal{G}_3^{(4)}(\rho_3)$ at $p_0 \leq p \leq 1$, the result is not convex in the full range of this region. Thus, we should choose

$$\rho_3(p) = \frac{p - p_0}{1 - p_0}|\Phi_3\rangle\langle\Phi_3| + \frac{1-p}{1-p_0}\rho_3(p_0) \tag{4.9}$$

as an optimal decomposition, which simply results in the right-hand side of Eq. (2.11) with $p_0 = 3/5$.

V. CONCLUSION

	$\mathcal{C}$ (concurrence)	τ (three-tangle)
ρ_1	$\frac{1}{2}\left(1-2\sqrt{p}-p\right)\theta(\alpha_1-p) \quad \left(\alpha_1=(\sqrt{2}-1)^2\right)$	0
ρ_2	$\left(\frac{3-p}{6}-\frac{\sqrt{2}}{3}\sqrt{p(3-p)}\right)\theta(\alpha_2-p) \quad \left(\alpha_2=\frac{1}{3}\right)$	?
ρ_3	$\mathcal{C}_{AB}=\frac{1}{2}\left(1-2\sqrt{p}-p\right)\theta(\alpha_1-p)$ $\mathcal{C}_{AC}=\mathcal{C}_{AD}=\mathcal{C}_{BC}=\mathcal{C}_{BD}$ $=\frac{1}{2}\left(1-p-\sqrt{p(2-p)}\right)\theta(\alpha_3-p) \quad \left(\alpha_3=\frac{2-\sqrt{2}}{2}\right)$ $\mathcal{C}_{CD}=\frac{1}{2}\left\{1-\sqrt{\frac{p}{2}}\left(\sqrt{1+\sqrt{p(2-p)}}+\sqrt{1-\sqrt{p(2-p)}}\right)\right\}$	$\tau_{ACD}=\tau_{BCD}=0$ $\tau_{ABC}=\tau_{ABD}=?$

Table III:Entanglement for sub-states of ρ_j $(j=1,2,3)$.

We compute the three-kinds of true four-way entanglement measures $\mathcal{F}_j^{(4)}$ $(j=1,2,3)$ and their corresponding linear entanglement monotones $\mathcal{G}_j^{(4)}$ $(j=1,2,3)$ analytically for four-qubit rank-2 mixed states $\rho_j=p|\Phi_j\rangle\langle\Phi_j|+(1-p)|\mathrm{W}_4\rangle\langle\mathrm{W}_4|$ $(j=1,2,3)$. All optimal decompositions consist of 2, 3, 4, and 5 vectors.

Our results can be used to find many different mixed states, which have vanishing entanglement. For example, Let us consider $\mathcal{F}_1^{(4)}$ with p_0 in Eq. (3.4). Let us represent, for simplicity, $|\Phi_2\rangle$ and $|\mathrm{W}_4\rangle$ as

$$|\Phi_2\rangle=\begin{pmatrix}1\\0\end{pmatrix} \qquad |\mathrm{W}_4\rangle=\begin{pmatrix}0\\1\end{pmatrix}. \tag{5.1}$$

Imagine the two-dimensional space spanned by $|\Phi_2\rangle$ and $|\mathrm{W}_4\rangle$ represented by a Bloch sphere. Then, the states in the Bloch sphere can be expressed as $\rho=\frac{1}{2}(\mathbb{1}+\boldsymbol{r}\cdot\boldsymbol{\sigma})$, where $|\boldsymbol{r}|=1$ and $|\boldsymbol{r}|<1$ denote the pure and mixed states, respectively. In this representation the Bloch vectors of $|\Phi_2\rangle$, $|\mathrm{W}_4\rangle$, and $|Z_2(p_0,\varphi)\rangle$ are

$$\begin{gathered}\boldsymbol{r}(\Phi_2)=(0,0,1) \quad \boldsymbol{r}(\mathrm{W}_4)=(0,0,-1)\\ \boldsymbol{r}(Z_2(p_0,\varphi))=(-2\sqrt{p_0(1-p_0)}\cos\varphi,-2\sqrt{p_0(1-p_0)}\sin\varphi,2p_0-1).\end{gathered} \tag{5.2}$$

Thus, any states located in the tetrahedron , whose vertices are $(0,0,-1)$, $(-2\sqrt{p_0(1-p_0)},0,2p_0-1)$, $(\sqrt{p_0(1-p_0)},-\sqrt{3p_0(1-p_0)},2p_0-1)$, and $(\sqrt{p_0(1-p_0)},\sqrt{3p_0(1-p_0)},2p_0-1)$ in the Bloch sphere representation, have vanishing $\mathcal{F}_1^{(4)}$ and $\mathcal{G}_1^{(4)}$.

One can use our results to discuss the monogamy properties[29] of entanglement. For this purpose, however, we should compute the entanglement for the sub-states of ρ_j $(j = 1, 2, 3)$. The entanglement of the sub-states is summarized at Table III. As this table shows, some three-tangle, at least for us, cannot be computed analytically. This is because still we do not have a closed formula for computing the three-tangles.

As far as we know, this is a first paper, which derives the the entanglement of four-qubit mixed states analytically. Thus, our result may serve as a quantitative reference for future studies of entanglement in quadripartite and/or multipartite mixed states.

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[1] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, England, 2000).

[2] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Quantum Entanglement*, Rev. Mod. Phys. **81** (2009) 865 [quant-ph/0702225] and references therein.

[3] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres and W. K. Wootters, *Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channles*, Phys.Rev. Lett. **70** (1993) 1895.

[4] C. H. Bennett and S. J. Wiesner, *Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states*, Phys. Rev. Lett. **69** (1992) 2881.

[5] V. Scarani, S. Lblisdir, N. Gisin and A. Acin, *Quantum cloning*, Rev. Mod. Phys. **77** (2005) 1225 [quant-ph/0511088] and references therein.

[6] A. K. Ekert , *Quantum Cryptography Based on Bells Theorem*, Phys. Rev. Lett. **67** (1991) 661.

[7] C. Kollmitzer and M. Pivk, Applied Quantum Cryptography (Springer, Heidelberg, Germany, 2010).

[8] T. D. Ladd, F. Jelezko, R. Laflamme, Y. Nakamura, C. Monroe, and J. L. O'Brien, *Quantum Computers*, Nature, **464** (2010) 45 [arXiv:1009.2267 (quant-ph)].

[9] G. Vidal, *Efficient classical simulation of slightly entangled quantum computations*, Phys. Rev.

Lett. **91** (2003) 147902 [quant-ph/0301063].

[10] C. H. Bennett, D. P. DiVincenzo, J. A. Smokin and W. K. Wootters, *Mixed-state entanglement and quantum error correction*, Phys. Rev. **A 54** (1996) 3824 [quant-ph/9604024].

[11] V. Vedral, M. B. Plenio, M. A. Rippin and P. L. Knight, *Quantifying Entanglement*, Phys. Rev. Lett. **78** (1997) 2275 [quant-ph/9702027].

[12] V. Vedral and M. B. Plenio, *Entanglement measures and purification procedures*, Phys. Rev. **A 57** (1998) 1619 [quant-ph/9707035].

[13] S. Hill and W. K. Wootters, *Entanglement of a Pair of Quantum Bits*, Phys. Rev. Lett. **78** (1997) 5022 [quant-ph/9703041]; W. K. Wootters, *Entanglement of Formation of an Arbitrary State of Two Qubits*, ibid. **80** (1998) 2245 [quant-ph/9709029].

[14] J. A. Smolin, *Four-party unlockable bound entangled state*, Phys. Rev. **A 63** (2001) 032306 [quant-ph/0001001]; S. Ghosh, G. Kar, A. Roy, A. Sen(De), and U. Sen, *Distinguishability of Bell States*, Phys. Rev. Lett. **87** (2001) 277902 [quant-ph/0106148]; Y. X. Chen, J. S. Jin, and D. Yang, *Distillation of multiple copies of Bell states*, Phys. Rev. **A 67** (2003) 014302; D. Yang and Y. X. Chen, *Mixture of multiple copies of maximally entangled states is quasipure*, Phys. Rev. **A 69** (2004) 024302.

[15] A. Miranowicz and S. Ishizaka, *Closed formula for the relative entropy of entanglement*, Phys. Rev. **A78** (2008) 032310 [arXiv:0805.3134 (quant-ph)]; H. Kim, M. R. Hwang, E. Jung and D. K. Park, *Difficulties in analytic computation for relative entropy of entanglement*, ibid. **A 81** (2010) 052325 [arXiv:1002.4695 (quant-ph)]; D. K. Park, Relative entropy of entanglement for two-qubit state with z-directional Bloch vectors, Int. J. Quant. Inf. **8** (2010) 869 [arXiv:1005.4777 (quant-ph)]; S. Friedland and G Gour, *Closed formula for the relative entropy of entanglement in all dimensions*, J. Math. Phys. **52** (2011) 052201 [arXiv:1007.4544 (quant-ph)]; M. W. Girard, G. Gour, and S. Friedland, *On convex optimization problems in quantum information theory*, arXiv:1402.0034 (quant-ph); E. Jung and D. K. Park, *REE From EoF* arXiv:1404.7708 (quant-ph).

[16] C. H. Bennett, S. Popescu, D. Rohrlich, J. A. Smolin, and A. V. Thapliyal, *Exact and asymptotic measures of multipartite pure-state entanglement*, Phys. Rev. **A 63** (2000) 012307 [quant-ph/9908073].

[17] W. Dür, G. Vidal and J. I. Cirac, *Three qubits can be entangled in two inequivalent ways*, Phys.Rev. **A 62** (2000) 062314 [quant-ph/0005115].

[18] A. Acín, D. Bruß, M. Lewenstein, and A. Sanpera, Classification of Mixed Three-Qubit States, Phys. Rev. Lett. **87** (2001) 040401 [quant-ph/0103025].

[19] F. Verstraete, J. Dehaene, and D. De Moor, *Normal forms and entanglement measures for multipartite quantum states*, Phys. Rev. **A 68** (2003) 012103 [quant-ph/0105090].

[20] V. Coffman, J. Kundu and W. K. Wootters, *Distributed entanglement*, Phys. Rev. **A 61** (2000) 052306 [quant-ph/9907047].

[21] A. Uhlmann, *Fidelity and concurrence of conjugate states*, Phys. Rev. **A 62** (2000) 032307 [quant-ph/9909060].

[22] R. Lohmayer, A. Osterloh, J. Siewert and A. Uhlmann, *Entangled Three-Qubit States without Concurrence and Three-Tangle*, Phys. Rev. Lett. **97** (2006) 260502 [quant-ph/0606071]; C. Eltschka, A. Osterloh, J. Siewert and A. Uhlmann, *Three-tangle for mixtures of generalized GHZ and generalized W states*, New J. Phys. **10** (2008) 043014 [arXiv:0711.4477 (quant-ph)]; E. Jung, M. R. Hwang, D. K. Park and J. W. Son, *Three-tangle for Rank-*3 *Mixed States: Mixture of Greenberger-Horne-Zeilinger, W and flipped W states*, Phys. Rev. **A 79** (2009) 024306 [arXiv:0810.5403 (quant-ph)]; E. Jung, D. K. Park, and J. W. Son, *Three-tangle does not properly quantify tripartite entanglement for Greenberger-Horne-Zeilinger-type state*, Phys. Rev. **A 80** (2009) 010301(R) [arXiv:0901.2620 (quant-ph)]; E. Jung, M. R. Hwang, D. K. Park, and S. Tamaryan, *Three-Party Entanglement in Tripartite Teleportation Scheme through Noisy Channels*, Quant. Inf. Comp. **10** (2010) 0377 [arXiv:0904.2807 (quant-ph)].

[23] C. Eltschka and J. Siewert, *Entanglement of Three-Qubit Greenberger-Horne-Zeilinger-Symmetric States*, Phys. Rev. Lett. **108** (2012) 020502 [arXiv:1304.6095 (quant-ph)].

[24] J. Siewert and C. Eltschka, *Quantifying Tripartite Entanglement of Three-Qubit Generalized Werner States*, Phys. Rev. Lett. **108** (2012) 230502.

[25] A. Osterloh and J. Siewert, *Constructing N-qubit entanglement monotones from antilinear operators*, Phys. Rev. **A 72** (2005) 012337 [quant-ph/0410102]; D. Ž. Đoković and A. Osterloh, *On polynomial invariants of several qubits*, J. Math. Phys. **50** (2009) 033509 [arXiv:0804.1661 (quant-ph)]; A. Osterloh and J. Siewert, *The invariant-comb approach and its relation to the balancedness of multiple entangled states*, New J. Phys. **12** (2010) 075025 [arXiv:0908.3818 (quant-ph)].

[26] A. Osterloh and J. Siewert, *Entanglement monotones and maximally entangled states in multipartite qubit systems*, Quant. Inf. Comput. **4** (2006) 0531 [quant-ph/0506073].

[27] F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, *Four qubits can be entangled in nine different ways*, Phys. Rev. **A 65** (2002) 052112.

[28] A. Osterloh, J. Siewert and A. Uhlmann. *Tangles of superpositions and the convex-roof extension*, Phys. Rev. **A 77** (2008) 032310 [arXiv:0710.5909 (quant-ph)].

[29] C. Eltschka and J. Siewert, *Monogamy equalities for qubit entanglement from Lorentz invariance*, arXiv:1407.8195 (quant-ph).